Heating of the Beurling operator and the estimates of its norm

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Abstract

We establish a new estimate for the norm of the Beurling transform $T\varphi(z) := \frac{1}{\pi}\int\int \frac{\varphi(\zeta)dA(\zeta)}{(\zeta-z)^2}$ in $L^p(dA)$. Namely, we prove $\|T\|_{L^p\to L^p} \le 2(p-1)$ for all $p \ge 2$. The method of Bellman function is used; however, we were unable to find the exact Bellman function of the problem. Instead, we use a certain approximation to the Bellman function, which leads to the factor 2 (instead of conjectural 1).

Notation

$:=$	equal by definition;
$x := (x_1, x_2)$	;
$D(x, R)$	the disk centered at x of radius R;
$k(x,t) := \frac{1}{4\pi t}e^{-\frac{\|x\|^2}{4t}}$	the heat kernel on the plane;
$\mathcal{D}$	a collection of dyadic intervals

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0 Introduction: main objects and results

The main object of this note is the Beurling operator given by :

$$T\varphi(z) := \frac{1}{\pi}\int\int \frac{\varphi(\zeta)dA(\zeta)}{(\zeta - z)^2}\,.$$

Here dA denotes area Lebesgue measure on $\mathbb{C}$. The goal of this paper is to present a new estimate of the norm of T. This estimate falls short of the proof of a wellknown conjecture saying that

$$\|T\|_{L^p\to L^p} = p-1,\ \ p\geq 2\,. \tag{0.1}$$

Here we prove that $\|T\|_{L^p\to L^p} \leq 2(p-1)$ for all $p\geq 2$, which is two times worse than (0.1). However, this is two times better than the previous estimate $\|T\|_{L^p\to L^p} \leq 4(p-1)$ established in [3]. But this is still two times worse than the conjectured sharp estimate by $p-1$. After the first preprint version of the present paper appeared, Rodrigo Bañuelos and Pedro Mendez-Hernandez [9] informed us that they also managed to improve 4 to 2 by modifying the methods from [3].

Actually this problem has a long history, and has been reappearing in many papers on the regularity of quasiconformal homeomorphisms and quasiregular maps. The L^p@-theory of quasiregular mappings was essentially started with the works of B. Bojarski [5]-[6]. Later this subject came under intensive investigation. In particular, the best integrability of K@-quasiconformal mappings and the (in a sense dual, see [28]) problem of the minimal regularity of the quasiregular mappings were discussed in many papers. We mention here [18], [15]-[17], [19], [22], [23], [24], [27], [28]. The best integrability result was established finally in [2]. The best minimal regularity result was obtained recently in [30]. The method of [30] will be applied in the present note to establish the inequality

$$\|T\|_{L^p\to L^p} \leq 2(p-1),\ \ p\geq 2\,. \tag{0.2}$$

By the same method, it is possible to prove that

$$\|\left(\Sigma_{j,k=1}^2\left|\frac{\partial^2 f}{\partial x_j\partial x_k}\right|^2\right)^{1/2}\|_p \leq \sqrt{2}\,p\,\|\Delta f\|_p,\ \ f\in W_2^p,\ p\geq 2\,, \tag{0.3}$$

which is better than in [24].

1 Consequences of the "$p-1$ estimate"

Let us formulate the analytic and geometric consequences of getting rid of 2 in (0.2). We are dealing with (local) solutions of Beltrami equation

$$f_{\bar z} - \mu f_z = 0\,. \tag{1.1}$$

We ask two questions.

1) Suppose $\|\mu\|_\infty = k < 1$. If a solution is *a priori* in W^2_1 locally, what is the ensured smoothness of this local solution? It is classical that f must belong to $W_1^{2+\varepsilon(k)}$ locally, where $\varepsilon(k) > 0$. The best $\varepsilon(k)$ was the essence of the problem by F. Gehring solved by K. Astala [2]. The best $\varepsilon(k)$ turned out to be equal to $\frac{1-k}{k}$. It is not attainable in general.

2) Suppose $\|\mu\|_\infty = k < 1$. If a solution is *a priori* in W^q_1 locally (now $q < 2$) , what is smallest q that ensures $f \in W^2_1$ locally (and then by [2] ensures $f \in W_1^{1+1/k-\tau}$ for any positive τ). The smallest q turns out to be $1 + k$. It is attainable (see [30]).

The two questions are intimately related to estimate (0.1). We explain the reason for that. Consider (1.1) in a neighborhood W of the origin, and put $V = \frac{1}{2}W$. Let φ be a C^∞ function supported on W and equal to 1 on V. We put $g := \varphi f_{\bar z}$ and consider $f - \frac{1}{\pi}\int_{\mathbb{C}} \frac{g(\zeta)}{z-\zeta} dA(\zeta)$. Here dA stands for planar Lebesgue measure. The disdtributional operator $\bar\partial$ of the latter expression vanishes in V. So, we have a function h analytic in V such that $f = h + \frac{1}{\pi}\int_{\mathbb{C}} \frac{g(\zeta)}{z-\zeta} dA(\zeta)$. Consequently,

$$f_z = h' + Tg\,.$$

in V. If we multiply (1.1) by φ, use the notations $g := \varphi f_{\bar z}$ and the previous expression for f_z, we get in V

$$g - \mu\varphi Tg = r := \mu\varphi h'\,.$$

On $U := \frac{1}{2}V$ the function r is bounded and so it is in any $L^p(U, dA)$. We denote by M the operator of multiplication by $\mu\varphi$ in $L^p(U, dA)$, $\|M\| \le k$. We denote by $t(p) := \|T\|_{L^p(U,dA)\to L^p(U,dA)}$ and consider the identity

$$(I - MT)g = r$$

to conclude that the inequality

$$kt(p) < 1 \tag{1.2}$$

implies the inclusion $g \in L^p$ in U, which is the same as to say that $f \in W_1^p$ locally.
Now we see that (0.1) would imply that for every p with

$$p < 1 + \frac{1}{k}$$

the solution in question of (1.1) is in $L^p(dA)$ locally.

The above considerations yield also **a lower estimate on the norm of T on L^p**. This reasoning is borrowed from [19]. Suppose that $t(p)$ is strictly smaller than $p-1$. Using (1.2) in te same sense as above, we see that if $\|\mu\|_\infty = k < 1$, then solution of (1.1) (that is *a priori* in W_1^2 locally) is in fact in $W_1^{1+1/k+\varepsilon}$ for some $\varepsilon > 0$. But it is easy to compute that the function $f(z) := z|z|^{-\frac{2k}{1+k}}$ satisfies (1.1) with $\mu(z) = -k\frac{z}{\bar{z}}$. So, the L^∞-norm of μ is k. However, f is not in $W_1^{1+1/k+\varepsilon}$ near the origin. It is not even in $W_1^{1+1/k}$. In fact, it is readily computable that $|f_z| = C(k)|z|^{-\frac{2k}{1+k}}$, which does not belong to any $L^p(dA)$ for p greater or equal to $1+1/k$. Thus, $\|T\|_p \geq p-1$.

We think that we have presented enough motivation for our interest in the estimation of the L^p-norm of such a particular Fourier multiplier as T and of related to it multipliers to be considered here.

2 Littlewood-Paley identity for heat extensions

For the Beurling operator T we can write the identity $T = R_1^2 - R_2^2 + 2iR_1R_2$, R_i are the planar Riesz transforms.

We fix, say, R_1^2 and two complex valued test functions $\varphi, \psi \in C_0^\infty$. We use heat extensions. For a function f on the plane, its heat extension is given by the formula

$$f(y,t) := \frac{1}{4\pi\, t} \iint_{\mathbb{R}^2} f(x) \exp(-\frac{|x-y|^2}{4t})\, dx_1 dx_2, \ \ (y,t) \in \mathbb{R}^3_+ \,.$$

We usually employ the same letter to denote a function and its heat extension.

Lemma 2.1. *Let $\varphi, \psi \in C_0^\infty$. Then the integral $\iiint \frac{\partial\varphi}{\partial x_1} \cdot \frac{\partial\psi}{\partial x_1} \, dx_1 dx_2 dt$ converges absolutely and*

$$\iint R_1^2 \varphi \cdot \psi \, dx_1 dx_2 = -2 \iiint \frac{\partial\varphi}{\partial x_1} \cdot \frac{\partial\psi}{\partial x_1} \, dx_1 dx_2 dt \,. \tag{2.1}$$

Proof. Actually, the proof of this lemma is trivial. It is based on the fact that a function is an integral of its derivative, and also involves Parseval's formula. Consider complex valued functions $\varphi, \psi \in C_0^\infty$ and write

$$\iint \psi R_1^2 \varphi dx_1 dx_2 = \iint \frac{\xi_1^2}{\xi_1^2 + \xi_2^2} \hat{\varphi}(\xi_1, \xi_2) \hat{\psi}(-\xi_1, -\xi_2) d\xi_1 d\xi_2$$

$$= 2 \iint \int_0^\infty e^{-2t(\xi_1^2+\xi_2^2)} \xi_1^2 \hat{\varphi}(\xi_1, \xi_2) \hat{\psi}(-\xi_1, -\xi_2) dt d\xi_1 d\xi_2$$

$$= -2 \int_0^\infty \iint i\xi_1 \hat{\varphi}(\xi_1, \xi_2) e^{-t(\xi_1^2+\xi_2^2)} \cdot i\xi_1 \hat{\psi}(-\xi_1, -\xi_2) e^{-t(\xi_1^2+\xi_2^2)} d\xi_1 d\xi_2 dt$$

$$= 2 \int_0^\infty \iint \frac{\partial\varphi}{\partial x_1}(x_1, x_2, t) \frac{\partial\psi}{\partial x_1}(x_1, x_2, t) dx_1 dx_2 dt$$

$$= 2 \iiint_{\mathbb{R}^3_+} \frac{\partial\varphi}{\partial x_1}(x_1, x_2, t) \frac{\partial\psi}{\partial x_1}(x_1, x_2, t) dx_1 dx_2 dt \,.$$

We have used Parseval's formula twice, and also the absolute convergence of the integrals $\iiint_{\mathbb{R}^3_+} e^{-2t(\xi_1^2+\xi_2^2)} \xi_1^2 \hat{\varphi}(\xi_1, \xi_2) \hat{\psi}(\xi_1, \xi_2) d\xi_1 d\xi_2 dt$, $\iiint_{\mathbb{R}^3_+} \frac{\partial\varphi}{\partial x_1}(x_1, x_2, t) \frac{\partial\psi}{\partial x_1}(x_1, x_2, t) dx_1 dx_2 dt$. For the first integral this is obvious and easy for the second one. We leave this as an exercise for the reader. □

3 The Bellman function proof of (0.2)

We warn the reader that sometimes it will be convenient to think that $\mathbb{C}$ is $\mathbb{R}^2$, and that the absolute value $|\cdot|$ is the norm of a vector in $\mathbb{R}^2$, $\|\cdot\|$.

Let φ, ψ be complex valued test functions in $C_0^\infty(\mathbb{R}^2)$. We denote their heat extensions to $\mathbb{R}^3_+$ by the same letters and use Lemma 2.1. Then we will see that estimates of combinations of $\langle R_i^2 \varphi, \psi \rangle$ is reduced to estimate of integrals we see in the next theorem. Notice also that if U_ρ denotes the operator $U_\rho \varphi(z) := f(e^{i\rho} z)$, then $2R_1 R_2 = U^*_{\pi/4}(R_1^2 - R_2^2) U_{\pi/4}$. Therefore, the proof of (0.2) follows immediately from Theorem 3.1 and Lemma 2.1.

Theorem 3.1. *For any complex valued* $\varphi, \psi \in C_0^\infty$ *we have*

$$2\iiint_{\mathbb{R}^3_+} \left|\frac{\partial\varphi}{\partial x_1}\right| \left|\frac{\partial\psi}{\partial x_1}\right| dx_1\, dx_2\, dt + 2\iiint_{\mathbb{R}^3_+} \left|\frac{\partial\varphi}{\partial x_2}\right| \left|\frac{\partial\psi}{\partial x_2}\right| dx_1\, dx_2\, dt$$

$$\leq (p-1)\|\varphi\|_p \|\psi\|_q\,.$$

In particular,

$$\|R_1^2 - R_2^2\|_p \leq p-1\,,\ \ \|2R_1R_2\|_p \leq p-1 \ \text{ for all } p\,, 2 \leq p < \infty\,.$$

In the proof of Theorem 3.1 we use the following key result. (In what follows, $d^2 f$ denotes the Hessian form that is the second differential form of f.)

Theorem 3.2. *For any* $p \geq 2$ *define the domain* $D_p := \{0 < (X, Y, \xi, \eta) \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 : \|\xi\|^p < X,\ \|\eta\|^q < Y, \}$. *Let* K *be any compact subset of* D_p, *and let* ε *be an arbitrary positive number. Then there exists a function* $B = B_{\varepsilon,p,K}(X, Y, x, y)$ *infinitely differentiable in a small neighbourhood of* K *such that*

1) $0 \leq B \leq (1+\varepsilon)(p-1)X^{1/p}Y^{1/q}$,

2) $-d^2 B \geq 2\|d\xi\|\|d\eta\|$.

We prove Theorem 3.2 later. Now we use it to obtain the proof of Theorem 3.1.

Proof. Consider two functions $\varphi, \psi \in C_0^\infty$. Now take $B = B_{\varepsilon,p,K}$, where a compact K remains to be chosen.

We are interested in

$$b(x,t) := B(|\varphi|^p(x,t), |\psi|^q(x,t), \varphi(x,t), \psi(x,t))\,.$$

This is a well defined function, because Cauchy inequality ensures that the 4-vector v,

$$v := (|\varphi|^p(x,t), |\psi|^q(x,t), \varphi(x,t), \psi(x,t))$$

lies in D_p for any $(x,t) \in \mathbb{R}^3_+$. Also we can fix any compact subset M of the open set $\mathbb{R}^3_+$ and guarantee that for $(x,t) \in M$, the vector v lies in a compact K. In fact, notice that for compactly supported φ, ψ the mapping $(x,t) \to v(x,t)$ maps compacts in $\mathbb{R}^3_+$ to compacts in D_p. Now just take K

large enough according to M which will run (in our future considerations) over larger and larger compacts in $\mathbb{R}^3_+$.

We want to apply Green's formula to $b(x,t)$. To do this, we introduce a Green's function $G(x,t)$ as in [13]. Let us consider a sufficiently large cylinder $\Omega := \Omega_l := D(0,l) \times (0,l)$. Let $\partial'\Omega = \partial D(0,l) \times (0,l)$. Consider the following Green's function.

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + \Delta\right) G_\Omega = -\delta_{0,1} \text{ in } \Omega\,, \\ \qquad\qquad\quad G_\Omega = 0 \text{ on } \partial'\Omega\,, \\ \qquad\qquad\quad G_\Omega = 0 \text{ when } t = l\,. \end{cases}$$

Here $\delta_{0,1}$ is a δ function at the point $(0,1)$.

Let $k(x,t) := \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t}}$, $x := (x_1, x_2)$ be a heat kernel in $\mathbb{R}^3_+$. One can understand $k(0,t)$ as the temperature of point $(0,0)$ on the plain at time $t > 0$ if initially (at $t = 0$) one has delta distribution of temperature placed at $(0,0)$. It is important to keep in mind that

$$G_\Omega(0,0) \to k(0,1)\,, \text{ when } l \to \infty\,. \tag{3.1}$$

In fact, just compare the interpretation of $k(0,1)$ with the fact that $G_\Omega(0,0)$ is temperature at time 1 when the same initial distribution is given but when also the temperature on $\partial'\Omega_l$ is kept to be 0. However, when l is large, it is clear that these two quantities are very close.

We also need a Green's function in the cylinder $\Omega(R,R^2) = D(0,lR) \times (0,lR^2)$:

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + \Delta\right) G^R_\Omega = -\delta_{0,R^2} \text{ in } \Omega(R,R^2) = D(0,lR) \times (0,lR^2)\,, \\ \qquad\qquad\quad G^R_\Omega = 0 \text{ on } \partial'\Omega(R,R^2) = \partial D(0,lR) \times (0,lR^2)\,, \\ \qquad\qquad\quad G^R_\Omega = 0 \text{ when } t = lR^2\,. \end{cases}$$

One can easily see that the following holds:

Lemma 3.3.

$$G^R_\Omega(x,t) = \frac{1}{R^2} G_\Omega\left(\frac{x}{R}, \frac{t}{R^2}\right).$$

We are ready to apply Green's formula to $b(x,t)$.

Let us first estimate $b(0,R^2) = B(|\varphi|^p(0,R^2), \cdots, \psi(0,R^2))$. Using the first property of Theorem 3.2, we get ($x = (x_1,x_2)$ as always, and $1/p + 1/q = 1$):

$$b(0,R^2) \le (1+\varepsilon)(p-1)(|\varphi|^p(0,R^2))^{1/p}(|\psi|^q(0,R^2))^{1/q}\,.$$

Thus,

$$b(0,R^2) \le (1+\varepsilon)(p-1)\left(\frac{1}{4\pi R^2}\iint |\varphi|^p(x)e^{-\frac{|x|^2}{4R^2}}\right)^{1/p}\cdot$$
$$\cdot\left(\frac{1}{4\pi R^2}\iint |\psi|^q(x)e^{-\frac{|x|^2}{4R^2}}\right)^{1/q}.$$

Now by Green's formula in $C(R,R^2)$:

$$\begin{aligned}
b(0,R^2) &= -\iiint_{\Omega(R,R^2)\cap\{t>\delta\}}, b(x,t)\left(\frac{\partial}{\partial t}+\Delta\right)G_\Omega^R(x,t)\ dx_1dx_2dt\\
&= \iiint_{\Omega(R,R^2)\cap\{t>\delta\}} G_\Omega^R(x,t)\left(\frac{\partial}{\partial t}-\Delta\right)b(x,t)\,dx_1dx_2dt\\
&+ \iint_{D(0,R)} b(x,\delta)G_\Omega^R(x,\delta)dx_1dx_2\\
&+ \iint_{\partial'\Omega(R,R^2)\cap\{t>\delta\}}\left(\frac{\partial b}{\partial n_{\text{outer}}}G_\Omega^R-\frac{\partial G_\Omega^R}{\partial n_{\text{outer}}}b\right)dsdt\\
&= \iiint_{\Omega(R,R^2)\cap\{t>\delta\}} G_\Omega^R(x,t)\left(\frac{\partial}{\partial t}-\Delta\right)b(x,t)\,dx_1dx_2dt\\
&+ \iint_{D(0,lR)} b(x,\delta)G_\Omega^R(x,\delta)dx_1dx_2+\iint_{\partial'\Omega(R,R^2)\cap\{t>\delta\}}\frac{\partial G_\Omega^R}{\partial n_{\text{inner}}}b\,dsdt\\
&\ge \iiint_{\Omega(R,R^2)\cap\{t>\delta\}} G_\Omega^R(x,t)\left(\frac{\partial}{\partial t}-\Delta\right)b(x,t)\,dx_1dx_2dt\,.
\end{aligned}$$

The last inequality is clear: the double integrals are both nonnegative, because b is nonnegative and because G_Ω^R is nonnegative and vanishes on the side boundary. Let us combine estimates of $b(0,R^2)$ into ($\Omega_{R,\delta} := \Omega(R,R^2)\cap\{t>\delta\}$)

$$\iiint_{\Omega_{R,\varepsilon}} G_\Omega^R(x,t)\left(\frac{\partial}{\partial t}-\Delta\right)b(x,t) \le$$

$$\frac{(1+\varepsilon)(p-1)}{4\pi R^2}\left(\iint |\varphi|^p(x)e^{-\frac{|x|^2}{4R^2}}\right)^{1/p}\left(\iint |\psi|^q(x)e^{-\frac{|x|^2}{4R^2}}\right)^{1/q}. \quad (3.2)$$

Fix R and $\delta > 0$ and choose the compact set $M = \{(x,t) \colon x \in \mathrm{clos}(D(0,lR)), \delta \le t \le lR^2\}$. Vector function v maps M to a compact subset of the domain D_p. Let it be called K. Choose $B = B_{\varepsilon,p,K}$ from Theorem 3.2.
The next calculation is simple but it is key to the proof. In it

$$v = (|\varphi|^p(x,t), |\psi|^q(x,t), \varphi(x,t), \psi(x,t)) .$$

Lemma 3.4. *If* $(x,t) \in M$ *then*

$$\left(\frac{\partial}{\partial t} - \Delta\right) b(x,t) = \left((-d^2B)\,\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_1}\right)_{\mathbb{R}^6} + \left((-d^2B)\,\frac{\partial v}{\partial x_2}, \frac{\partial v}{\partial x_2}\right)_{\mathbb{R}^6} .$$

Proof.

$$\frac{\partial}{\partial t} b = (\nabla B, \frac{\partial v}{\partial t})_{\mathbb{R}^6} ,$$

$$\Delta b = \left((d^2B)\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_1}\right)_{\mathbb{R}^6} + \left((d^2B)\frac{\partial v}{\partial x_2}, \frac{\partial v}{\partial x_2}\right)_{\mathbb{R}^6} + (\nabla B, \Delta v)_{\mathbb{R}^6} .$$

We just used the chain rule. Now

$$\left(\frac{\partial}{\partial t} - \Delta\right) b = (\nabla B, (\frac{\partial v}{\partial t} - \Delta v)_{\mathbb{R}^6} - \left((d^2B)\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_1}\right)_{\mathbb{R}^6} - \left((d^2B)\frac{\partial v}{\partial x_2}, \frac{\partial v}{\partial x_2}\right)_{\mathbb{R}^6} .$$

However, the first term is zero because all entries of the vector v are solutions of the heat equation. □

By Theorem 3.2 in M

$$-d^2B(X,Y,\xi,\eta) \ge 2\|d\xi\|\|d\eta\| . \tag{3.3}$$

For $(x,t) \in M$ Lemma 3.4 gives:

$$\left(\frac{\partial}{\partial t} - \Delta\right) b(x,t) \ge 2\left|\frac{\partial \varphi}{\partial x_1}\right|\left|\frac{\partial \psi}{\partial x_1}\right| + \left|\frac{\partial \varphi}{\partial x_2}\right|\left|\frac{\partial \psi}{\partial x_2}\right| . \tag{3.4}$$

Combining (3.2), (3.4) we get

$$2\iiint_M G^R_\Omega(x,t)\left(\left|\frac{\partial \varphi}{\partial x_1}\right|\left|\frac{\partial \psi}{\partial x_1}\right| + \left|\frac{\partial \varphi}{\partial x_2}\right|\left|\frac{\partial \psi}{\partial x_2}\right|\right) \le \frac{(1+\varepsilon)(p-1)}{4\pi R^2}(\iint |\varphi|^p(x))^{1/p}(\iint |\psi|^q(x))^{1/q} . \tag{3.5}$$

Now it is time to use Lemma 3.3. So (3.5) implies

$$2\iiint_M G_\Omega\Big(\frac{x}{R},\frac{t}{R^2}\Big)\left(\left|\frac{\partial\varphi}{\partial x_1}\right|\left|\frac{\partial\psi}{\partial x_1}\right|+\left|\frac{\partial\varphi}{\partial x_2}\right|\left|\frac{\partial\psi}{\partial x_2}\right|\right)$$

$$\le \frac{(1+\varepsilon)(p-1)}{4\pi}\,\Big(\iint|\varphi|^p(x)\Big)^{1/p}\Big(\iint|\psi|^q(x)\Big)^{1/q}. \quad (3.6)$$

But $M = \{(x,t) : x \in \mathrm{clos}(D(0,R)),\, \delta \le t \le R^2\}$. Let us fix any compact M_0 in $\mathbb{R}^3_+$ and choose R and $\delta > 0$ in such a way that $M_0 \subset M$. Restrict the integration in (3.6) to M_0 and make $R \to \infty$. Taking into account that

$$G_\Omega(x/R, t/R^2) \to G_\Omega(0,0)\,,$$

we obtain

$$2G_{\Omega_l}(0,0)\iiint_{M_0}\left(\left|\frac{\partial\varphi}{\partial x_1}\right|\left|\frac{\partial\psi}{\partial x_1}\right|+\left|\frac{\partial\varphi}{\partial x_2}\right|\left|\frac{\partial\psi}{\partial x_2}\right|\right)$$

$$\le \frac{(1+\varepsilon)(p-1)}{4\pi}\,\Big(\iint|\varphi|^p(x)\Big)^{1/p}\Big(\iint|\psi|^q(x)\Big)^{1/q}. \quad (3.7)$$

Now it is time to tend $\Omega = \Omega_l$ to $\mathbb{R}^3_+$ by making l to go to infinity. By (3.1) we conclude that $G_{\Omega_l}(0,0) \to \frac{1}{4\pi}$. So (3.7) becomes

$$2\iiint_{M_0}\left(\left|\frac{\partial\varphi}{\partial x_1}\right|\left|\frac{\partial\psi}{\partial x_1}\right|+\left|\frac{\partial\varphi}{\partial x_2}\right|\left|\frac{\partial\psi}{\partial x_2}\right|\right)$$

$$\le (1+\varepsilon)(p-1)\,\Big(\iint|\varphi|^p(x)\Big)^{1/p}\Big(\iint|\psi|^q(x)\Big)^{1/q}. \quad (3.8)$$

But M_0 is an arbitrary compact set in the upper half space and ε is an arbitrary positive number. Therefore,

$$2\iiint_{\mathbb{R}^3_+}\left(\left|\frac{\partial\varphi}{\partial x_1}\right|\left|\frac{\partial\psi}{\partial x_1}\right|+\left|\frac{\partial\varphi}{\partial x_2}\right|\left|\frac{\partial\psi}{\partial x_2}\right|\right)$$

$$\le (p-1)\,\Big(\iint|\varphi|^p(x)\Big)^{1/p}\Big(\iint|\psi|^q(x)\Big)^{1/q}. \quad (3.9)$$

This proves Theorem 3.1. □

4 The existence of Bellman function. Proof of Theorem 3.2

We start with a simple "model" operator T_σ. To define it, let $\mathcal{D}$ denote a family of dyadic intervals on the line. To each $I \in \mathcal{D}$ we assign its Haar function: $h_I = 1/\sqrt{|I|}$ on I_+ and $h_I = -1/\sqrt{|I|}$ on I_-, I_+ and I_- being right and left halves of I respectively. Every nice complex-valued function (continuous with compact support on one of I's, say on $[0,1]$) can be written as its Haar series: $f = \Sigma_I (f, h_I) h_I$. Consider the operator $T_\sigma f = \Sigma_I \sigma_I (f, h_I) h_I$, where σ_I is any sequence of complex numbers with absolute values 1. Notations: wee use $\langle f \rangle_I$ to denote $\frac{1}{|I|}\int_I f dx$.

The logic will be the following. We want to get a sharp estimate of $\|T_\sigma\|_{L^p \to L^p}$ via p. This problem has been solved by Burkholder. He found out in [10] that ($p \geq 2$)

$$\sup_\sigma \|T_\sigma\|_p \leq p - 1\,. \tag{4.1}$$

He proved (4.1) by constructing a certain function of two real variables (actually another Bellman function) which has certain convexity and size property. The reader is referred to the papers of Burkholder [10], [11], or the book of D. Stroock [32] to study his approach. In particular, in [32], page 344, it is written about (4.1): "Quite recently Burkholder has discovered *the right argument*:... it is completely elementary. Unfortunately, it is also completely opaque. Indeed, his new argument is nothing but an elementary verification that he has got the right answer; it gives no hint about how he came to that answer". Further on "for those who want to know the secret behind his proof, Burkholder has written an explanation in his article" [10]. Here is Burkholder's function ($p \geq 2$)

$$b(x,y) = (|x| - (p-1)|y|)(|x| + |y|)^{p-1}\,.$$

Actually, stochastic Bellman PDE explains readily the way to write this function, and this is made, for example, in [33].

We want to use this Bellman function of Burkholder in our problem. But we are unable to do that. The reason is simple. Burkholder's function variables stand for certain martingales, which in his case are related to each other: one is subordinate to the other. In our case we replace this variables not by martingales but by functions: the first is $R_1^2\varphi$, the second is φ, where φ is a test function. There is no subordination here. The only differential relationship between this two functions (actually between their heat extensions) is the following

$$\frac{\partial}{\partial t} R_1^2 \varphi = \frac{\partial^2}{\partial x_1^2} \varphi \,.$$

This is a differential relation of the second order, and being such it has no connection with the subordination property, which interplayed so essentially with a very special convexity property of Burkholder function (see [10]). It would be connected with subordination property (and then with convexity) would it be a differential relation of the first order. What we mean can be illustrated by this oversimplified example. It is obvious that superposition of convex function a with linear function l is convex, but superposition of convex a and convex l may not be convex ($a(x) = e^{-x}, l(x) = x^2$). That is exactly the obstacle to use Burkholder's function and compose it with our second order Riesz transforms.

We do not see the way to win over this difficulty. We prefer another approach. It follows the approach in [30].

Idea: we formulate Burkholder's inequality in equivalent form (just in its dual form). The resulting inequality generates another Bellman function. This will be our B from Theorem 3.2.

As we already said we will use the following lemma due to Burkholder:

Lemma 4.1. *Let H be a separable Hilbert space. Let (X_n, F_n, P) and (Y_n, F_n, P) are H-valued martingales. If for almost every ω*

$$\|X_0(\omega)\|_H \leq \|Y_0(\omega)\|_H, \;\; \|X_n(\omega) - X_{n-1}(\omega)\|_H \leq \|Y_n(\omega) - Y_{n-1}(\omega)\|_H, \;\forall n,$$

then, for each $p \in (1, \infty)$,

$$\|X_n\|_{L^p(P,H)} \leq \max(p-1, 1/(p-1)) \|Y_n\|_{L^p(P,H)} \,.$$

From the lemma one can easily obtain the following theorem.

Theorem 4.2. *Let $J \in \mathcal{D}$, $f \in L^p(J), g \in L^{p'}(J)$. Let $p \geq 2$. Then*

$$\frac{1}{4} \frac{1}{|J|} \Sigma_{I \in \mathcal{D}, I \subset J} |\langle f \rangle_{I_+} - \langle f \rangle_{I_-}| |\langle g \rangle_{I_+} - \langle g \rangle_{I_-}| |I| \leq (p-1) \langle |f|^p \rangle_J^{1/p} \langle |g|^{p'} \rangle_J^{1/p'} \,.$$

Proof. Without loss of generality $J = [0, 1]$. Let F_n be σ-algebra generated by all dyadic subintervals of J with length at least 2^{-n}. Consider $\omega \in [0, 1]$ and put

$$Y_n(\omega) := \Sigma_{I \subset J, |I| \geq 2^{-n}} (f, h_I) h_I(\omega) \,.$$

Fix any sequence of complex numbers $\sigma_I = e^{i\alpha_I}$, $\alpha_I \in \mathbb{R}$ and consider

$$X_n(\omega) := \Sigma_{I\subset J, |I|\geq 2^{-n}} \sigma_I (f, h_I) h_I(\omega) \,.$$

Both (X_n, F_n, dx) and (X_n, F_n, dx) are martingales. Clearly $Y_n - Y_{n-1} = \Sigma_{I\subset J, |I|=2^{-n}} (f, h_I) h_I$, also $X_n - X_{n-1} := \Sigma_{I\subset J, |I|=2^{-n}} \sigma_I (f, h_I) h_I$. And these martingales satisfy the assumptions of Burkholder's lemma. The Hilbert space $H = \mathbb{R}^2$ is naturally identified with $\mathbb{C}$. Now $\|f\|_{L^p} = \lim_{n\to\infty} \|Y_n\|_{L^p(H)}$, $\|T_\sigma f\|_{L^p} = \lim_{n\to\infty} \|X_n\|_{L^p(H)}$; Burkholder's lemma implies

$$\|T_\sigma f\|_{L^p} \leq (p-1)\|f\|_{L^p} \tag{4.2}$$

for $p \geq 2$ and for any sequence σ as above.

Let us reformulate (4.2) as $|(T_\sigma f, g)| \leq (p-1)\|f\|_{L^p}\|g\|_{L^{p'}}$. Definition of T_σ now implies

$$\forall \sigma_I = e^{i\alpha_I}, \frac{1}{|J|} |\Sigma_I \sigma_I (f, h_I)\overline{(g, h_I)}| \leq (p-1) \langle |f|^p\rangle_J^{1/p} \langle |g|^{p'}\rangle_J^{1/p'} \,. \tag{4.3}$$

Notice that $(f, h_I) = \frac{1}{2}\sqrt{|I|}(\langle f\rangle_{I_+} - \langle f\rangle_{I_-})$. We also have the right to vary σ_I. So the theorem follows.

□

Theorem 4.3. *In the domain $G = \{(\Phi, \Psi, \phi, \psi) \in \mathbb{R}\times\mathbb{R}\times\mathbb{C}\times\mathbb{C} : |\phi|^p < \Phi, |\psi|^{p'} < \Psi\}$ there exists a function $B(\Phi, \Psi, \phi, \psi)$ such that for any 4-tuples $a = (\Phi, \Psi, \phi, \psi)$, $a_- = (\Phi_-, \Psi_-, \phi_-, \psi_-)$, $a_+ = (\Phi_+, \Psi_+, \phi_+, \psi_+)$ such that $a = \frac{a_- + a_+}{2}$ the following holds*

$$B(a) - \frac{1}{2}(B(a_-) + B(a_+)) \geq \frac{1}{4}|\phi_- - \phi_+||\psi_- - \psi_+| \,.$$

Also everywhere in G

$$0 \leq B(a) \leq (p-1)\Phi^{1/p}\Psi^{1/p'} \,.$$

For every compact subset of K of G one can find infinitely smooth functions B_K on K such that the first estimate holds. Consider $\phi = \xi_1 + i\eta_1, \psi = \xi_2 + i\eta_2$, where $\xi' s$ and $\eta' s$ are real. If we consider B_K as function of 6 real variables, then we can consider its 6×6 Jacobi matrix and corresponding second differential form–Hessian. Then the Hessian of B_K must satisfy the following inequality

$$-d^2 B_K \geq 2|d\phi||d\psi| = 2((d\xi_1)^2 + (d\eta_1)^2)^{1/2}((d\xi_2)^2 + (d\eta_2)^2)^{1/2} \,.$$

At the same time, given a positive ε, B_K can be chosen to satisfy

$$0 \leq B_K(a) \leq (1+\varepsilon)(p-1)\Phi^{1/p}\Psi^{1/p'} \,.$$

Proof. Fix $(\Phi, \Psi, \phi, \psi) \in G$. Consider all complex-valued functions f, g on J such that $\Phi = \langle |f|^p \rangle_J$, $\Psi = \langle |g|^{p'} \rangle_J$, $\phi = \langle f \rangle_J$, $\psi = \langle g \rangle_J$. Let

$$B(\Phi, \Psi, \phi, \psi) := \sup\{\frac{1}{4}\frac{1}{|J|}\Sigma_{I\in\mathcal{D},\, I\subset J} |\langle f\rangle_{I_+} - \langle f\rangle_{I_-}||\langle g\rangle_{I_+} - \langle g\rangle_{I_-}||I|\}\,,$$

where supremum is taken over all such f, g. This supremum does not depend on the interval J. This observation helps to prove the first inequality $B(a) - \frac{1}{2}(B(a_-)+B(a_+)) \geq \frac{1}{4}|\phi_- - \phi_+||\psi_- - \psi_+|$ exactly as it has been done in any paper on Bellman functions. On the other hand, the second inequality, $0 \leq B(a) \leq (p-1)\Phi^{1/p}\Psi^{1/p'}$ is proved already—this is the claim of Theorem 4.2.

If we fix a compact K, we can also fix a very small ε (much smaller than the distance from K to the boundary of G and we can consider $\frac{1}{\varepsilon^6}S(\frac{a}{\varepsilon})$, where S is a C_0^∞ function supported by the unit ball of $\mathbb{R}\times\mathbb{R}\times\mathbb{C}\times\mathbb{C} = \mathbb{R}^6$ centered at zero. Then just mollify B by convoluting it with $\frac{1}{\varepsilon^6}S(\frac{a}{\varepsilon})$. The concavity inequality will hold with no change for the new function. The size inequality can become $(1 + C_K\varepsilon)$ times worse. □

References

[1] K. Astala, T. Iwaniec, E. Saksman, *Beltrami operators*, Preprint 210, Jyvaskyla University, May 1999.

[2] K. Astala, *Area distortion of quasiconformal mappings*, Acta Math., **173**, (1994), 37-60

[3] R. Bañuelos, G. Wang, *Sharp inequalities for martingales with applications to the Beurling-Ahlfors and Riesz transforms*, Duke Math. J., **80**, No. 3, (1995), 575-600.

[4] R. Bañuelos, P. Méndez-Hernández: *Sharp inequalities Riesz transforms and space time Brownian motion*, To appear in Indiana Univ. Math. J.

[5] B. V. Bojarski, *Homeomorphic solutions of Beltrami systems*, Dokl. Akad. Nauk. SSSR, **102**, (1955), 661-664.

[6] B. V. Bojarski, *Generalized solutions of a system of differential equations of first order and elliptic type with discontinuous coefficients*, Mat. Sbornik, No 43, **85**, (1957), 451-503.

[7] B. V. Bojarski, *Quasiconformal mappings and general structure properties of systems of noninear equations elliptic in the sense of Lavrentiev*, Symposia Mathematica (1976).

[8] B. V. Bojarski T. Iwaniec, *Quasiconformal mappings and and non-linear elliptic equations in two variables I,II*, Bull. Acad. Pol. Sci., **22**, No. 5, (1974), 473-484.

[9] R. Bañuelos, P. Méndez-Hernández: *Sharp inequalities for heat kernels of Schrödinger operators and applications to spectral gaps*, J. Funct. Anal. **176** (2000), no. 2, 368–399.

[10] D. L. Burkholder, *Explorations in martingale theory and its applications*, Ecole d'Eté de Probabilité de Saint-Flour XIX–1989, 1-66, Lecture Notes in Mathematics, **1464**, Springer, Berlin, 1991.

[11] D. L. Burkholder, *Boundary value problems and sharp inequalities for martingale transforms,* Ann. Prob., **12**(1984), 647–802.

[12] St. Buckley *Estimates for operator norms on weighted spaces and reverse Jensen inequalities*, Trans. Amer. Math. Soc., **340**, (1993), 253-273.

[13] R. Fefferman, C. Kenig, J. Pipher *The theory of weights and the Dirichlet problem for elliptic equations*, Annals of Math., **134**, (1991), 65-124.

[14] J. Garcia-Cuerva, J. Rubio de Francia *Weighted Norm Inequalities And Related Topics*, North-Holland Mathematics Studies, 116, North-Holland, Amsterdam-New York-Oxford, 1985.

[15] F. W. Gehring, *Open problems*, Proceedings of Roumanian-Finnish Seminar on Teichmuller Spaces and Quasiconformal Mappings, 1969, page 306.

[16] F. W. Gehring, *The L^p–integrability of the partial derivatives of a quasiconformal mapping*, Acta Math., **130** (1973), 265-277.

[17] F. W. Gehring, *Topics in quasiconformal mappings*, Proceedings of the ICM 1986, Berkeley, 62-80.

[18] F. W. Gehring, E. Reich *Area distortion under quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser AI, **388**, (1966), 1-15.

[19] T. Iwaniec *Extremal inequalities in Sobolev spaces and quasiconformal mappings*, Z. Anal. Anwendungen, **1**, (1982), 1-16.

[20] T. Iwaniec *The best constant in a BMO-inequality for the Beurling-Ahlfors transform*, Mich. Math. J., **33**, (1987), 387-394.

[21] T. Iwaniec *Hilbert transform in the complex plane and the area inequalities for certain quadratic differentials*, Mich. Math. J., **34**, (1987), 407-434.

[22] T. Iwaniec *L^p-theory of quasiregular mappings.* In: Quasiconformal space mappings, ed. Matti Vuorinen, Lecture Notes in Math., 1508, Springer 1992.

[23] T. Iwaniec, G. Martin, *Quasiregular mappings in even dimensions*, Acta Math., **170**, (1992), 29-81.

[24] T. Iwaniec, G. Martin, *Riesz transforms and related singular integrals*, J. Reine Angew. Math., **473**, (1996), 25-57.

[25] St. Petermichl, J. Wittwer, *A sharp weighted estimate on the norm of Hilbert transform via invariant A_2 characteristic of the weight*, Preprint, Michigan State University, 2000.

[26] J. Wittwer, , Thesis, University of Chicago, 2000.

[27] O. Lehto, *Quasiconformal mappings and singular integrals*, Sympos. Math., **XVIII**, (1976), 429-453, Academic Press, London.

[28] O. Lehto, K. Virtanen, *Quasiconformal mappings in the plane*, Springer-Verlag, Berlin-Heidelberg, 1973.

[29] F. Nazarov, S. Treil, and A. Volberg, *The Bellman functions and two-weght inequalities for Haar multipliers*, J. of the Amer. Math. Soc., v. 12, No. 4, 1999, 909-928.

[30] St. Petermichl, A. Volberg, *Heating of the Beurling operator: weakly quasiregular maps on the plane are quasiregular*, Preprint, Michigan State Unv., 2000, 1-22.

[31] E. Stein, *Harmonic Analysis: Real-variable Methods, Orthogonality, And Oscillatory Integrals*, Princeton Math. Series, 43, Monographs in Harmonic Analysis, Princeton Univ. Press, Princeton, NJ, 1993.

[32] D. W. Stroock: *Probability Theory, an Analytic View*, Cambridge University Press, 1993.

[33] A. Volberg: Bellman approach to some problems in harmonic analysis, Séminaire aux équations dérives partielles, n° XX, Ecole Polytéchnique, 2002.