Regular Inductive Limits of Locally Complete Spaces

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Abstract

An inductive limit $(E, \tau) = (E_n, \tau_n)$ is regular if and only if every sequence $(x_k) \in l_p(E, \sigma)$ belongs to $l_p(E_n, \sigma_n)$ for some $n \in \mathbb{N}$. A property $l_{p,q}(\sigma)$-retractivity is defined. Every regular (LF)-space is $l_{p,q}(\sigma)$-retractive. Finally, every locally complete inductive limit of locally complete spaces which satisfies Retakh's condition (M_0) is regular.

1 Introduction.

For a Hausdorff locally convex space (X, T) and for $1 \leq p < \infty$, the space of absolutely p-summable sequences is defined by

$$l_p(X,T) = \left\{ (x_k) \in X : \sum_k \rho^p(x_k) < \infty \right.$$
$$\left. \text{for every continuous seminorm in } (X,T) \right\}.$$

And the space of absolutely bounded sequences

$$l_\infty(X,T) = \left\{ (x_k) \in X : \sup_k \rho(x_k) < \infty \right.$$
$$\left. \text{for every continuous seminorm in } (X,T) \right\}.$$

In particular, if $T = \sigma$ is the weak topology, then every continuous seminorm $\rho(\cdot)$ is given by $|f(\cdot)|$, for some $f \in X'$.

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In [1] it was defined the concept of $l_{p,q}$-summability: a sequence $(x_k) \in (X,T)$ is $l_{p,q}$-summable if $(x_k)\in l_p(X,T)$ and for every $(a_k)\in l_q(\mathbb{C})$, $\frac{1}{p}+\frac{1}{q}=1$, we have $\sum_k a_k x_k \overset{(X,T)}{\rightarrow} x$. In [9], Qiu studied l_q-completeness: a locally convex space (X,T) is l_q-complete if every sequence $(x_k) \in l_p(X,T)$ is $l_{p,q}$-summable in (X,T); and he proved that for (every) $1 \leq q < \infty$, l_q-completeness is equivalent to local completeness.

Throughout the paper $\{(E_n, \tau_n)\}_n$ is a numerable inductive sequence of Hausdorff locally convex spaces and $(E,\tau) = \operatorname{ind}(E_n, \tau_n)$ or simply E its Hausdorff inductive limit. Recall that E is regular if every bounded subset in E is contained and bounded in one of the steps; and E is sequentially retractive if every null sequence in E converges to zero in some step.

We say E satisfies the Retakh's condition (M) (respectively (M_0)) (e.g., see [14]) if there exists an increasing sequence $(U_n)_n$ of absolutely convex neighbourhoods of zero, every U_n in (E_n, τ_n) with the following property: for every $n \in \mathbb{N}$ there is $m > n$ such that E and E_m induce the same (resp. weak) topology on U_n. We will assume that every such U_n is τ_n-closed and that for every $n \in \mathbb{N}$, (E_{n+1}, τ_{n+1}) and (E, τ) (resp. (E_{n+1}, σ_{n+1}) and (E, σ)) induce the same topology on U_n, which we do with out loss of generality. We have that (M) implies (M_0), but the converse does not always hold (see [13]). And we say E satisfies condition (Q) (see [14]) if the increasing condition for (M) is dropped.

Let us say that many authors have studied these conditions (M), (M_0) and Q. Among others: Vogt in [13] studied condition (M) for (LF)-spaces. He obtained several important results about them, *e.g.* that on (LF)-spaces condition (M) implies completeness, regularity and sequential retractivity. Recently, Wengenroth in [14] proved the following very important results on (LF)-spaces: condition (M), condition (Q), sequential retractivity and other stronger retractivity conditions are equivalent; and he solved the classical Grothendieck's problem on completeness of regular (LF)-spaces for the case of inductive limits of Fréchet-Montel spaces. Qiu, in [8] studied (LM)-spaces with property (M_0) and he obtained a number of equivalences for regularity.

In [6], Kucera proved that for (LF)-spaces regularity and sequential completeness are equivalent. Later, in [4], Kucera and Gómez-W. proved that regular inductive limits of sequentially complete spaces are sequentially complete; they also ask for the following question: If $(E,\tau) = \operatorname{ind}(E_n, \tau_n)$ is a sequentially complete inductive limit of sequentially complete spaces, when is (E,τ) regular?

Basically, this work is directed to present a partial answer to the Kucera-Gómez's question and a weak retractivity condition which is satisfied by every regular (LF)-space.

First, using a strong result of Qiu [10], we prove the following equivalence for regularity: an inductive limit $(E,\tau) = \operatorname{ind}(E_n, \tau_n)$ is regular if and only if for every sequence $(x_k) \in l_p(E,\sigma)$ there exists $n \in \mathbb{N}$ such that $(x_k) \in l_p(E_n, \sigma_n)$, $1 < p \leq \infty$. From this, we define a weak retractivity condition: an inductive limit $(E,\tau) = \operatorname{ind}(E_n, \tau_n)$ is $l_{p,q}(\sigma)$-retractive if every sequence $(x_k)_k$ which is $l_{p,q}$-summable in (E,σ) is also $l_{p,q}$-summable in some (E_n, σ_n).

It is a very well known fact that sequentially complete spaces are locally complete. Then for a more general context we will treat our original question with locally complete inductive limits of locally complete spaces. On this conditions regularity and $l_{p,q}(\sigma)$-retractivity are equivalent. As a particular case, every regular (LF)-space satisfies this weaker retractivity condition. Finally, we prove that a locally complete inductive limit of locally complete spaces, which satisfies condition (M_0) or is webbed is regular.

2 Regularity and local completeness.

In order to prove the first theorem, we make some easy observations:

1. A sequence $(x_k)_k \in (X,T)$ is bounded if and only if for every $f \in E'$ and for every $(a_k)_k \in l_1(\mathbb{C})$ we have $\sum_{\mathbb{N}} |a_k f(x_k)| < \infty$.

2. For an inductive limit $(E,\tau) = \operatorname{ind}(E_n, \tau_n)$ the following conditions are equivalent:

a) E is regular.

b) For every bounded sequence $(x_k)_k \in (E,\tau)$ there exists $n \in \mathbb{N}$ such that $(x_k)_k \in E_n$ and it is τ_n-bounded.

c) For every $(x_k)_k \in l_\infty(E,\tau)$ there exists $n \in \mathbb{N}$ such that for every $g \in E_n'$ and for every $(a_k)_k \in l_1(\mathbb{C})$ we have $\sum_{\mathbb{N}} |a_k g(x_k)| < \infty$.

Proposition 1 *An inductive limit $(E,\tau) = \operatorname{ind}(E_n, \tau_n)$ is regular if and only if for every $(x_k) \in l_p(E,\sigma)$ there exists $n \in \mathbb{N}$ such that $(x_k)_k \in l_p(E_n, \sigma_n)$, $1 < p < \infty$.*

Proof. In order to prove the sufficiency we will use observation (2.c). Let $(x_k)_k \in l_\infty(E,\tau)$, $f \in E'$ and $(a_k)_k \in l_1(\mathbb{C})$, then $\sum_{\mathbb{N}} |a_k f(x_k)| < \infty$. Now for every $k \in \mathbb{N}$, $a_k = |a_k|\, e^{i\theta_k} = |a_k|^{\frac{1}{p}}\, |a_k|^{\frac{1}{q}}\, e^{i\theta_k}$, $\frac{1}{p} + \frac{1}{q} = 1$. Then

$(b_k)_k = (|a_k|^{\frac{1}{q}}) \in l_q(\mathbb{C})$ and $(|a_k|^{\frac{1}{p}} e^{i\theta_k} x_k)$ is such that

$$\sum_{\mathbb{N}} \left| f(|a_k|^{\frac{1}{p}} e^{i\theta_k} x_k) \right|^p = \sum_{\mathbb{N}} |a_k| \, |f(x_k)|^p \leq R^p \sum_{\mathbb{N}} |a_k| < \infty,$$

that is $(|a_k|^{\frac{1}{p}} e^{i\theta_k} x_k)_k \in l_p(E, \sigma)$. Then we have that $(|a_k|^{\frac{1}{p}} e^{i\theta_k} x_k)_k \in l_p(E_n, \sigma_n)$ for some $n \in \mathbb{N}$. So, for every $g \in E'_n$ we have

$$\sum_{\mathbb{N}} \left| g(|a_k|^{\frac{1}{p}} e^{i\theta_k} x_k) \right|^p < \infty$$

and

$$\sum_{\mathbb{N}} |a_k g(x_k)| = \sum_{\mathbb{N}} |a_k|^{\frac{1}{p}} \, |a_k|^{\frac{1}{q}} \, |g(x_k)| = \sum_{\mathbb{N}} |b_k| \left| g(|a_k|^{\frac{1}{p}} e^{i\theta_k} x_k) \right|$$

$$\leq \|(b_k)_k\|_q \left\| (g(|a_k|^{\frac{1}{p}} e^{i\theta_k} x_k))_k \right\|_p < \infty.$$

Hence E is regular.

Suppose now that $(E, \tau) = \operatorname{ind}(E_n, \tau_n)$ is regular. Let $(x_k)_k \in l_p(E, \sigma)$, $f \in E'$ and $(a_k)_k \in l_q(\mathbb{C})$, $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\sum_{\mathbb{N}} |f(a_k x_k)| = \sum_{\mathbb{N}} |a_k f(x_k)| \leq \|(a_k)_k\|_q \, \|(f(x_k))_k\|_p < \infty.$$

So $\sum_{\mathbb{N}} a_k x_k$ is a weak unconditionally Cauchy series in E (see [10]). Hence by Theorem 1 in [10], there exists $n \in \mathbb{N}$ such that $\sum_{\mathbb{N}} a_k x_k$ is weak unconditionally Cauchy in E_n. So we have that for every $g \in E'_n$ and every $(a_k)_k \in l_q(\mathbb{C})$, $\frac{1}{p} + \frac{1}{q} = 1$, $\sum_{\mathbb{N}} |g(a_k x_k)| < \infty$. Now fix $g_0 \in E'_n$ and define the functional

$$\begin{aligned} T_{g_0} : \quad & l_q(\mathbb{C}) \to \mathbb{C} \\ & (a_k)_k \to \sum_{\mathbb{N}} a_k g_0(x_k) \end{aligned}$$

It is easy to see T_{g_0} is linear and continuous, since it is the limit of the partial sums which are linear and continuous. Hence for every $g_0 \in E'_n$, $(g(x_k))_k \in l_p(\mathbb{C})$, i.e. $(x_k)_k \in l_p(E_n, \sigma_n)$. ■

Now, we define a weak retractivity condition which is equivalent to regularity for many inductive limits.

Definition 2 *An inductive limit $(E, \tau) = \operatorname{ind}(E_n, \tau_n)$ is said to be $l_{p,q}(\sigma)$-retractive if every sequence $(x_k)_k$ which is $l_{p,q}$-summable in (E, σ) is $l_{p,q}$-summable in some (E_n, σ_n).*

In order to see this is well defined, note that if $(x_k)_k$ is an $l_{p,q}$-summable sequence in (E,σ) which is $l_{p,q}$-summable sequence in some (E_n,σ_n) and $(a_k)_k$ is a fixed sequence in $l_q(\mathbb{C})$, $\frac{1}{p}+\frac{1}{q}=1$, then $\sum_k a_k x_k \overset{(E,\sigma)}{\to} x_0$ and $\sum_k a_k x_k \overset{(E_n,\sigma_n)}{\to} z_0$. But, by the continuity of the following linear identity maps

$$(E_n, \sigma(E_n, E_n)) \overset{id}{\hookrightarrow} ind\,(E_n, \sigma(E_n, E_n)) \overset{id}{\hookrightarrow} (E, \sigma(E, E))$$

and since we consider only Hausdorff spaces, then $x_0 = z_0$.

Theorem 3 *Let every (E_n, τ_n) be locally complete and $(E,\tau) = \mathrm{ind}(E_n, \tau_n)$ be locally complete. (E,τ) is regular if and only if it is $l_{p,q}(\sigma)$-retractive.*

Proof. Suppose (E,τ) is regular. Let $(x_k)_k$ be $l_{p,q}$-summable in (E,σ), in particular $(x_k)_k \in l_p(E,\sigma)$. By proposition 1, $(x_k)_k \in l_p(E_n,\sigma_n)$ for some $n \in \mathbb{N}$. Since (E_n,σ_n) is locally complete, it follows that $(x_k)_k$ is $l_{p,q}$-summable in (E_n,σ_n). Conversely, suppose (E,τ) is $l_{p,q}(\sigma)$-retractive and let $(x_k)_k \in l_p(E,\sigma)$. Since (E,σ) is locally complete, it follows that $(x_k)_k$ is $l_{p,q}$-summable in (E,σ). Hence, $(x_k)_k$ is $l_{p,q}$-summable in some (E_n,σ_n), in particular $(x_k)_k \in l_p(E_n,\sigma_n)$. Hence, by proposition 1, (E,τ) is regular. ■

We will give now a result for webbed spaces. For general information about the basic properties of webs, we refer the reader to the works of De Wilde [2], Jarchow [5] and Robertson [11]. As an special remark, let us say that Valdivia [12] proved that locally complete webbed spaces are strictly webbed. And recall that for strictly webbed spaces we have the following classical result (e.g., see [10], lemma 1):

Lemma 4 *Let $(E,\tau) = \mathrm{ind}(E_n,\tau_n)$ be an inductive limit of strictly webbed spaces. If (E,τ) is locally complete then it is regular.*

Combining Theorem 3 and the last observations we have the following immediate corollary

Corollary 5 *Let $(E,\tau) = \mathrm{ind}(E_n,\tau_n)$ be an inductive limit of locally complete and webbed spaces. If (E,τ) is locally complete then it is regular and $l_{p,q}(\sigma)$-retractive.*

From lemma 4 it is easy to see another classical result: An (LF)-space is regular if and only if it is locally complete. Then applying Theorem 3 we conclude:

Corollary 6 *Every regular (LF)-space is $l_{p,q}(\sigma)$-retractive.*

In the next proposition we apply property (M_0) to obtain a result on regularity of locally complete spaces. Recall that a disk D in a Hausdorff locally convex space (X,T) is an absolutely convex, bounded and closed subset. Note before, that for a convex subset $G \subset E$, its closure satisfies $\overline{G}^{(E,\tau)} = \overline{G}^{(E,\sigma(E,E'))}$, then if the topologies are compatible it is not necessary to specify them when we take closure, and this is valid also for every duality invariant.

Proposition 7 *Let every (E_n,τ_n) be locally complete. If (E,τ)=ind(E_n,τ_n) is locally complete and it satisfies condition (M_0), then (E,τ) is regular.*

Proof. The proof is almost the same than that which appear in [3]. For the sake of completeness, the proof follows: It is sufficient to prove that every Banach disk $B \subset E$ is contained and bounded in some E_n. Let $B \subset E$ be a Banach disk. By [7] Proposition 8.5.20, there exists $p \in \mathbb{N}$, such that $B \subset p\overline{U_p}^E$. As a direct application of lemma 1 in [8], it follows that $\overline{U_p}^E = \bigcup_{k=p}^{\infty} \overline{U_p}^{E_k}$. Then $B \subset p\overline{U_p}^E = p \bigcup_{k=p}^{\infty} \overline{U_p}^{E_k}$.

Since B is τ-closed and τ-bounded, $B \cap E_k$ is τ_k-closed and $B \cap p\overline{U_p}^{E_k} \subset B$ is τ-bounded, for every $k \geq p$. Let $B_k = B \cap p\overline{U_p}^{E_k}$. We assume that every U_k is τ_k-closed, then $\frac{1}{p}B_k \subset \overline{U_p}^{E_k} \subset \overline{U_k}^{E_k} = U_k$, for every $k \geq p$. By condition (M_0), σ and σ_{k+1} coincide on U_k, then $\frac{1}{p}B_k$ is E_{k+1}-bounded. Now, local completeness of E_{k+1} implies that $\overline{B_k}^{E_{k+1}}$ is a Banach disk in E_{k+1}, so $(E_{\overline{B_k}^{E_{k+1}}}, \rho_{\overline{B_k}^{E_{k+1}}})$ is a Banach space continuously embedded in E_{k+1}.

Note that for every $k \geq p$,
$\overline{B_k}^{E_{k+1}} = \overline{B \cap p\overline{U_p}^{E_k}}^{E_{k+1}} \subset \overline{B \cap p\overline{U_p}^{E_{k+1}}}^{E_{k+2}} = \overline{B_{k+1}}^{E_{k+2}}$.

This implies that $\overline{B_k}^{E_{k+1}}$ is contained in $\overline{B_{k+1}}^{E_{k+2}} \cap E_{\overline{B_k}^{E_{k+1}}}$; therefore $(E_{\overline{B_k}^{E_{k+1}}}, \rho_{\overline{B_k}^{E_{k+1}}})$ is continuously embedded in $(E_{\overline{B_{k+1}}^{E_{k+2}}}, \rho_{\overline{B_{k+1}}^{E_{k+2}}})$.

It follows that ind$(E_{\overline{B_k}^{E_{k+1}}}, \rho_{\overline{B_k}^{E_{k+1}}})$ is an (LB)-space. In order to finish the proof, we will prove that this is a non proper (LB)-space. In other words, we will show that there exists $k_0 \in \mathbb{N}$ such that $(E_{\overline{B_{k_0}}^{E_{k_0+1}}}, \rho_{\overline{B_{k_0}}^{E_{k_0+1}}}) = (E_B, \rho_B)$:

Since B is τ-closed and $B_k \subset B$, we have $\overline{B_k}^{E_{k+1}} \subset B$. And $\overline{B_k}^{E_{k+1}} \subset B \cap E_{\overline{B_k}^{E_{k+1}}}$ which implies that the identity map $i_k : (E_{\overline{B_k}^{E_{k+1}}}, \rho_{\overline{B_k}^{E_{k+1}}}) \to (E_B, \rho_B)$ is continuous for every $k \geq p$.

On the other hand,

$$B = B \cap p \bigcup_{k=p}^{\infty} \overline{U_p}^{E_k} = \bigcup_{k=p}^{\infty} B \cap p\overline{U_p}^{E_k} = \bigcup_{k=p}^{\infty} B_k \subset \bigcup_{k=p}^{\infty} \overline{B_k}^{E_{k+1}} \subset B.$$

This means $span(B) = \bigcup_{k=p}^{\infty} span(\overline{B_k}^{E_{k+1}})$. Therefore the identity map

$i : \mathrm{ind}(E_{\overline{B_k}^{E_{k+1}}}, \rho_{\overline{B_k}^{E_{k+1}}}) \to (E_B, \rho_B)$ is continuous and onto. By the open-mapping theorem (see [7] Theorem 8.4.11) the inverse identity map

$j : (E_B, \rho_B) \to \mathrm{ind}(E_{\overline{B_k}^{E_{k+1}}}, \rho_{\overline{B_k}^{E_{k+1}}})$ is continuous. By Jarchow [5] Corollary 5.6.4, the space (E_B, ρ_B) is continuously embedded in some $(E_{\overline{B_{k_0}}^{E_{k_0+1}}}, \rho_{\overline{B_{k_0}}^{E_{k_0+1}}})$.

We conclude that B is contained and bounded in E_{k_0+1}. ■

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