

# Operadic Hochschild chain complex and free loop spaces

by

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**Abstract.** We construct, for any algebra  $A$  over an operad  $\mathcal{O}$ , an Hochschild chain complex,  $C_*^{\mathfrak{H}}(\mathcal{O}, A)$  which is also an  $\mathcal{O}$ -algebra. This Hochschild chain complex coincides with the usual one, whenever  $A$  is a commutative differential graded algebra. Let  $X$  is a simply connected space,  $N^*(-)$  be the singular cochain functor,  $X^{S^1}$  be the free loop space,  $\mathcal{C}_\infty$  be a cofibrant replacement of commutative operad and  $M_X$  a  $\mathcal{C}_\infty$ -cofibrant model of  $X$ . We prove that The operadic chain complex  $C_*^{\mathfrak{H}}(\mathcal{C}_\infty, M_X)$  is quasi-isomorphic to  $N^*(X^{S^1})$  as a  $\mathcal{C}_\infty$ -algebra. In particular, for any prime field of coefficients this identifies the action of the large Steenrod algebra on the Hochschild homology  $H_*^{\mathfrak{H}}(N^*(X))$  with the usual Steenrod operations on  $H^*(X^{S^1})$ .

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**Mathematics Subject Classifications(2000):** 55U35, 18Dxx, 55P15, 55P60

**Keywords:** Operads, Steenrod operations, Free loop space,  $E_\infty$ -algebra.

## Introduction.

As illustrated by the fundamental result of Mandell, [20],  $E_\infty$ -algebras are the good tools for the study of the homotopy theory of topological spaces. Indeed, for a prime field of coefficients the homology of a  $E_\infty$ -algebra is an unstable algebra over the large Steenrod algebra. This last property originates with the previous works of Dold, [9], May [21] and has been recently extensively studied by the first author and Livernet [8].

In this paper we develop an application of the homotopy theory of  $E_\infty$ -algebras to the study of the free loop space,  $X^{S^1}$  on a space  $X$ . For this purpose, we consider the almost free operad, denoted  $\mathcal{C}_\infty$ , which is a cofibrant model of the commutative operad. A quotient of this operad, also denoted  $\mathcal{C}_\infty$ , has been studied by Kadeisvili, [15], Kontsevich, [16] and Getzler-Jones, [10]. It appears that the category of  $\mathcal{C}_\infty$ -algebras is a closed model category which is very similar to the category of commutative and associative differential graded algebras.

It follows from Theorem C and Proposition 4.2-2:

**Main Theorem:** *We denote by  $N^*(-)$  the normalized singular cochain functor with coefficients in an arbitrary commutative ring  $\mathbb{k}$  and by  $H^*(-) =$*

$H(N^*(-))$  the functor of singular cohomology. Let  $X$  be a 1-connected space, if each  $H^i(X)$  is finitely generated then there exist natural equivalences of  $\mathcal{C}_\infty$ -algebras between  $N^*(X^{S^1})$  and the operadic Hochschild complex of the  $\mathcal{C}_\infty$ -algebra  $N^*(X)$ .

An associative  $\mathcal{C}_\infty$ -algebra is a particular case of strongly homotopy algebra (see 3.3). Our result is, in this special case, a substantial improvement of the results proved in [3] and [4] since the structure of unstable algebra over the Steenrod algebra but also all secondary cohomological operations are preserved [8].

All the paper is devoted to the proof of this result. The required knowledge about operads is presented in section 1. In section 2 we define the operad  $\mathcal{C}_\infty$  in relation with the Barratt-Eccles operad, extensively studied by Berger and Fresse, [1]. In section 3, we construct for any operad  $\mathcal{O}$  in the category of differential graded modules the operadic Hochschild complex of an  $\mathcal{O}$ -algebra,  $A$ . When  $A$  is supposed to be an associative and commutative differential graded algebra we compare the operadic Hochschild complex of an  $\mathcal{O}$ -algebra with the usual Hochschild complex. We also study the particular case when  $A$  is an almost free  $\mathcal{O}$ -algebra (Theorem B). In the last section we end the proof of our main result (Theorem C).

## 1. Backgrounds about algebras over an operad.

**1.1 Notation.** We denote by  $k\text{-GM}$  (resp.  $k\text{-DGM}$ ) the category of graded modules (resp. of differential graded modules). We also consider the forgetfull functor:

$$\# : k\text{-DGM} \rightarrow k\text{-GM}, \quad (V, d) \mapsto (V, d)_\# = V.$$

We are mainly concerned by the following categories :

$\Delta$ , the simplicial category of finite ordered sets with objects  $[n] = \{0, 1, \dots, n\}$  and non decreasing maps

$\mathbf{C}^\Delta$ , the category of cosimplicial objects and cosimplicial maps of  $\mathbf{C}$ :  
 $\underline{X} = (\{X^n\}_{n \geq 0}, d^i, s^i)$

$\mathbf{C}^{\Delta^{op}}$ , the category of simplicial objects and simplicial maps of  $\mathbf{C}$ :  $\underline{X} = (\{X_n\}_{n \geq 0}, d_i, s_i)$

**1.2 Operads.** Recall from [11], [10] and [17], that an operad  $\mathcal{O}$  is defined in any symmetric monoidal category  $\mathbf{C}$  as a sequence of left  $k[\Sigma_i]$ -modules

(where  $\Sigma_i$  is the symmetric group)  $\mathcal{O}(i), i \geq 0$ , with *composition products*

$$\begin{aligned} \mathcal{O}(n) \otimes \mathcal{O}(i_1) \otimes \mathcal{O}(i_2) \otimes \cdots \otimes \mathcal{O}(i_n) &\rightarrow \mathcal{O}(i_1 + i_2 + \cdots + i_n), \\ x_0 \otimes x_1 \otimes \cdots \otimes x_n &\mapsto x_0(x_1, x_2, \dots, x_n) \end{aligned}$$

which are equivariant, associative and with a unit. Homomorphisms of operads are defined in an obvious way.

The category of operads is a closed model category [2], [12], where the weak equivalences are the quasi-isomorphisms and the fibrations are the surjections.

The universal example of operad is the *endomorphism operad*. Let  $(V, d_V)$  be a differential graded module,  $\mathcal{E}nd_V$  is an operad in  $k\text{-DGM}$  such that:

$$\mathcal{E}nd_V(n) = \text{Hom}(V^{\otimes n}, V), n \geq 1.$$

**1.3 Algebras over an operad.** Let  $\mathcal{O}$  be an operad in  $k\text{-DGM}$ . An  $\mathcal{O}$ -algebra,  $(A, \rho)$ , is a differential graded module,  $A = (\{A_i\}_{i \in \mathbb{Z}}, d_A : A_i \rightarrow A_{i-1})$ , with an operadic representation

$$\rho : \mathcal{O} \rightarrow \mathcal{E}nd_A.$$

determined by a sequence of maps differential graded  $k$ -modules, called the *evaluation product*:

$$\tilde{\rho}_n : \mathcal{O}(n) \otimes A^{\otimes n} \rightarrow A, \quad \tilde{\rho}_n(x \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n) = \rho_n(x)(a_1 \otimes a_2 \otimes \cdots \otimes a_n).$$

invariant under the action of  $\Sigma_n$  and compatible with the composition product of  $\mathcal{O}$ .

$\mathcal{O}$ -algebras and homomorphisms of  $\mathcal{O}$ -algebras is a category, denoted  $\mathcal{O}\text{-ALG}$ . If  $\mathcal{O}'$  is another operad in  $k\text{-DGM}$ , a homomorphism of operads  $f : \mathcal{O} \rightarrow \mathcal{O}'$  induces a natural functor  $f^* : \mathcal{O}'\text{-ALG} \rightarrow \mathcal{O}\text{-ALG}$ .

The *free  $\mathcal{O}$ -algebra* generated by a differential graded module  $V$  is the differential graded module

$$F(\mathcal{O}, V) = \bigoplus_{k=0}^{\infty} \mathcal{O}(k) \otimes_{\Sigma_k} V^{\otimes k},$$

with evaluation products  $\tilde{\rho}_n : \mathcal{O}(n) \otimes F(\mathcal{O}, V)^{\otimes n} \rightarrow F(\mathcal{O}, V)$  induced by the composition products of  $\mathcal{O}$ . Any homomorphism  $f : V \rightarrow W$  of graded modules extends uniquely in homomorphism of graded modules  $F(\mathcal{O}, f) : F(\mathcal{O}, V) \rightarrow F(\mathcal{O}, W)$ . The functor  $F(\mathcal{O}, -) : k\text{-DGM} \rightarrow \mathcal{O}\text{-ALG}$  is a left adjoint to the forgetful functor  $\mathcal{O}\text{-ALG} \rightarrow k\text{-DGM}$ .

**1.4 The associative operad and the commutative operad.** For each  $n \geq 0$ , the canonical augmentation  $\epsilon_n : \mathbb{k}[\Sigma_n] \rightarrow \mathbb{k}$  of the group ring  $\mathbb{k}[\Sigma_n]$  defines a homomorphism of operads in  $\mathbb{k}$ -GM

$$\epsilon : \mathcal{A} \rightarrow \mathcal{C}$$

from the *associative operad*  $\mathcal{A}$  to the *commutative operad*  $\mathcal{C}$  such that  $\mathcal{A}(n) = \mathbb{k}[\Sigma_n]$  and  $\mathcal{C}(n) = \mathbb{k}$  with composition products given respectively by composition of permutations and multiplication in  $\mathbb{k}$ . An  $\mathcal{A}$ -algebra is an associative differential graded algebra while a  $\mathcal{C}$ -algebra is a commutative differential graded algebra which is also, associative. Indeed, the representations  $\rho : \mathcal{A} \rightarrow \mathcal{E}nd_A$  and  $\mathcal{C} \rightarrow \mathcal{E}nd_A$  are the iterated products:

$$\mathbb{k}[\Sigma_n] \otimes_{\Sigma_n} A^{\otimes n} = A^{\otimes n} \rightarrow A \text{ and } \mathbb{k} \otimes_{\Sigma_n} (A^{\otimes n}) = (A^{\otimes n})_{\Sigma_n} \rightarrow A.$$

The free  $\mathcal{C}$ -algebra (resp. a free  $\mathcal{A}$ -algebra) generated by the differential graded module  $V$  is the differential graded module  $F(\mathcal{C}, V) = \bigoplus_{n=0}^{\infty} \mathbb{k} \otimes_{\Sigma_n} V^{\otimes n} = \bigoplus_{n=0}^{\infty} (V^{\otimes n})_{\Sigma_n} = S(V)$  with graded commutative multiplication of the elements of  $V$  (resp.  $F(\mathcal{A}, V) = \bigoplus_{n=0}^{\infty} \mathbb{k}[\Sigma_n] \otimes_{\Sigma_n} V^{\otimes n} = \bigoplus_{n=0}^{\infty} V^{\otimes n} = T(V)$  the usual tensor algebra on  $V$ ).

**1.5** Let  $A$  be an  $\mathcal{O}$ -algebra. Then  $A$  is called *almost free* if  $A_{\#} = F(\mathcal{O}_{\#}, V)$  for a graded module  $V$ . If the category of  $\mathcal{O}$ -algebras is a closed model category (this is the case whenever  $\mathcal{O}$  is a cofibrant operad [2]) then any  $\mathcal{O}$ -algebra admits a cofibrant model. An  $\mathcal{O}$ -algebra is cofibrant if and only if it is a retract of an almost free  $\mathcal{O}$ -algebra.

## 2. The operads $\mathcal{BE}$ and $\mathcal{C}_{\infty}$ .

**2.1** Let us denote  $\underline{A} = (\{A_n\}_{n \in \mathbb{N}}, d_i, s_i)$  a simplicial differential graded module with internal differential  $d_A : A_{n,q} \rightarrow A_{n,q-1}$ . Then the *total complex* of the bicomplex

$$\underline{A}_{p-1,q} \xleftarrow{\sum (-1)^i d_i} \underline{A}_{p,q} \xrightarrow{d_A} \underline{A}_{p,q+1}$$

is denoted by  $\text{Tot}_*(\underline{A})$ :

$$\begin{aligned} \text{Tot}(\underline{A}) &= (\{\text{Tot}_n(\underline{A})\}_{n \in \mathbb{Z}}, d) \quad d : \text{Tot}_n(\underline{A}) \rightarrow \{\text{Tot}_{n-1}(\underline{A}) \\ \text{Tot}_n(\underline{A}) &= \bigoplus_{p+q=n} \underline{A}_{p,q} \quad dx = d_A x + (-1)^p \sum (-1)^i d_i x, x \in \underline{A}_{p,q}. \end{aligned}$$

Let  $D_n(\underline{A})$  be the subcomplex generated by degeneracies in  $\underline{A}_n$ . The quotient complex  $\text{Tot}_*(\underline{A})/D_*(\underline{A}) := N_*(\underline{A})$  is the *normalized differential*

*graded module*. The quotient map  $\text{Tot}(\underline{A}) \rightarrow N_*(\underline{A})$  is a chain equivalence, [19]-Theorem 6.1.

The singular chain complex of  $X$  with coefficients in  $\mathbb{k}$  is  $C_*(X; \mathbb{k}) := C_*(S_*(X; \mathbb{k}))$  and the normalized chain complex is  $N_*(X; \mathbb{k}) := N_*(S_*(X))$ . The subcomplex of normalized cochain complex  $N^*(X, \mathbb{k}) \subset \text{Hom}(C_*(X, \mathbb{Z}), \mathbb{k})$  is, in this paper, simply denoted  $N^*(X)$ . It follows from, [19]-theorem 9.1 that  $N^*$  is a contravariant functor from the category of pointed topological spaces to the category of augmented associative differential graded algebras.

The functor  $N(-)$

$$\mathbb{k}\text{-DGM}^{\Delta^{op}} \rightarrow \mathbb{k}\text{-DGM}$$

is a monoidal functor. The functor  $N$  transforms operads  $\mathcal{O}$  in  $\mathbb{k}\text{-DGM}^{\Delta^{op}}$  (resp.  $\mathcal{O}$ -algebras in  $\mathbb{k}\text{-DGM}^{\Delta^{op}}$ ) into operads in  $\mathbb{k}\text{-DGM}$  (resp. algebras over an operad in  $\mathbb{k}\text{-DGM}$ ), [17]-page 51. For further use we need the slightly more general result.

**2.2 Lemma.** *Let  $\underline{A} = \{A_{p,q}\}_{p \in \mathbb{N}, q \in \mathbb{Z}}$  be a simplicial  $\mathcal{O}$ -algebra. Then the total complex  $\text{Tot}(\underline{A})$  is an  $\mathcal{O}$ -algebra. Moreover,  $N_*(\underline{A})$  is an  $\mathcal{O}$ -algebra and the quotient map  $\text{Tot}(\underline{A}) \rightarrow N_*(\underline{A})$  is an equivalence of  $\mathcal{O}$ -algebras.*

**Proof.**

Let  $\underline{A}$  and  $\underline{B}$  be two simplicial graded modules and consider the *shuffle product*

$$sh : C_p \underline{A} \otimes C_q \underline{B} \rightarrow C_{p+q}(\underline{A} \times \underline{B}), a \otimes b \mapsto \sum_{\mu, \nu} (-1)^{\epsilon(\mu)} s_\nu a \times s_\mu b, \begin{cases} a \in A_p \\ b \in B_q \end{cases}$$

when the sum is taken over the  $p + q$  shuffles  $\mu_1 < \mu_2 < \dots < \mu_p, \nu_1 < \nu_2 < \dots < \nu_q$ ,  $\mu_i, \nu_j \in \{1, 2, \dots, p + q\}$ ,  $\epsilon(\mu)$  is the graded signature of the  $(p, q)$ -shuffle,  $s_\mu = s_{\mu_p} \circ s_{\mu_{p-1}} \circ \dots \circ s_{\mu_1}$ ,  $s_\nu = s_{\nu_q} \circ s_{\nu_{q-1}} \circ \dots \circ s_{\nu_1}$ , [19]-Chapter 8. If we assume that  $\underline{A}$  and  $\underline{B}$  are two simplicial differential graded modules, one easily check that  $sh$  commutes with the differentials so that we obtain:

$$sh : \text{Tot} \underline{A} \otimes \text{Tot} \underline{B} \rightarrow \text{Tot}(\underline{A} \times \underline{B}), \text{ and } sh : N_p \underline{A} \otimes N_q \underline{B} \rightarrow N_{p+q}(\underline{A} \times \underline{B}).$$

Since, when  $A = B$ , the shuffle product is associative (and commutative) one defines the *iterated shuffle product*

$$sh^0 = id, sh^{k+1} = (sh \otimes id) \circ sh^k, \quad k \geq 0$$

Let  $\tilde{\rho}_{n,k} : \mathcal{O}(k) \otimes (\underline{A}_n)^{\otimes k} \rightarrow A$  be an evaluation product of  $\underline{A}_n$ . We consider  $\mathcal{O}$  as a constant simplicial module and we define the map  $\hat{\rho}_k : \mathcal{O}(k) \otimes (N_*(\underline{A}))^{\otimes k} \rightarrow N_*(\underline{A})$  as the composite

$$\begin{array}{ccc} \mathcal{O}(k) \otimes (N_*(\underline{A}))^{\otimes k} & = & N_*(\mathcal{O}(k)) \otimes N_*(\underline{A})^{\otimes k} \\ & & \downarrow id \otimes sh^k \\ & & N_*(\mathcal{O}(k)) \otimes N_*(\underline{A}^{\otimes k}) \\ & & \downarrow sh \\ N_*(\underline{A}) & \xleftarrow{N_*(\tilde{\rho}_k)} & N_*(\mathcal{O}(k)) \otimes \underline{A}^{\otimes k} \end{array} .$$

It is then easy to check that the  $\hat{\rho}_n$  are composition products. □

**2.3 The operad  $\mathcal{BE}$ .** The operad  $\mathcal{BE}$  (also called the Barrat-Eccles operad, [1]-1.1.) is an operad in the category of differential graded  $\mathbb{k}$ -modules such that:

$$\mathcal{BE}(n) = N_*(W(\Sigma_n)) = \text{the normalized bar construction on } \Sigma_n.$$

**2.4 Some properties of  $\mathcal{BE}$ .**

1) The operad  $\mathcal{BE}$  is a resolution of the operad  $\mathcal{C}$ : the homomorphism of operads  $\bar{\epsilon} : \mathcal{BE} \xrightarrow{\sim} \mathcal{C}$  is defined by the augmentations of the bar resolution for each component.

2)  $\mathcal{BE}$  is an  $E_\infty$ -operad. Recall that an operad  $\mathcal{O} = \{(\mathcal{O}(n))_i\}_{i \geq 0}$  in  $\mathbb{k}$ -**DGM** is an  $E_\infty$ -operad if each  $\mathcal{O}(n)$  is an acyclic  $\Sigma_n$ -free module.

3) The natural map  $\epsilon : \mathcal{A} \rightarrow \mathcal{C}$  factorises as  $\mathcal{A} \rightarrow \mathcal{BE} \xrightarrow{\bar{\epsilon}} \mathcal{C}$ . In particular,  $\mathcal{BE}$ -algebras are associative algebras.

4) The operad  $\mathcal{BE}$  is not cofibrant.

5) Berger and Fresse [1] have proved that the normalized singular cochains  $N^*(-)$  is a functor from the category of topological spaces to the category of  $\mathcal{BE}$ -differential algebras.

6) If  $\mathbb{k} = \mathbb{F}_p$  is the prime field of characteristic  $p$  and  $A$  is a  $\mathcal{BE}$ -algebras then  $H(A)$  is an unstable algebra over the big Steenrod algebra ([21], [8]). Indeed, consider the Standard small free resolution  $\mathcal{W}$  (resp.  $\mathcal{W}'$  of the cyclic group of order  $p$ ;  $\pi \subset \Sigma_p$  (resp. of  $\Sigma_p$ ). Thus we obtain the homomorphism  $\mathcal{W} \rightarrow \mathcal{W}' \rightarrow \mathcal{BE}(p)$  and the evaluation product  $\tilde{\rho}_p : \mathcal{BE}(p) \otimes A^p \rightarrow A$  restricts to the structure map  $\mathcal{W} \otimes A^{\otimes p} \rightarrow A$  considered by May, [21], in order to define ‘‘algebraic Steenrod operations’’. Recall that the big Steenrod algebra, denoted  $B_p$ , is such that the quotient  $B_p/(P^0 = id)$  is the usual Steenrod algebra, see [20]-theorem 1.4. In particular, Adem relations are satisfied ([21] and [8]).

**2.5 The operad  $\mathcal{C}_\infty$ .** Let  $\mathcal{C}_\infty$  be a cofibrant replacement of  $\mathcal{C}$ . There exists a quasi-isomorphism of operads

$$\mathcal{C}_\infty \rightarrow \mathcal{BE}.$$

In particular, by remarks 5) and 6) above,  $N^*(X)$  is a  $\mathcal{C}_\infty$ -algebra and any quasi-isomorphism of  $\mathcal{C}_\infty$ -algebras  $A \rightarrow N^*(X)$  identifies the action of the large Steenrod operations on  $H(A)$  to the usual action on  $N^*(X)$ .

By (1.5) any  $\mathcal{C}_\infty$ -algebra admits a cofibrant replacement which is an almost free  $\mathcal{C}_\infty$ -algebra, see [13] and [2]

### 3. The Operadic Hochschild chain complex.

**3.1** Let us recall that the category of  $\mathcal{O}$ -algebras has all limits and colimits, [10]-Theorem 1.13. In particular, [20]-3, the coproduct of two  $\mathcal{O}$ -algebras  $A$  and  $B$  is an  $\mathcal{O}$ -algebra, denoted  $A \amalg B$ . Hereafter, we will use the following notation:  $A \xrightarrow{l} A \amalg B \xrightarrow{r} B$  for the natural inclusions and  $A \amalg A \xrightarrow{\nabla} A$  for the folding map. The symmetric group  $\Sigma_n$  acts on  $A \amalg^n = A \amalg \cdots \amalg A$  ( $n$  terms) by permutations of factors. We denote by  $\tau_n : A \amalg^n \rightarrow A \amalg^n$  the homomorphism corresponding to the cyclic permutation  $(n, 1, 2, \dots, n-1) \in \Sigma_n$ .

**3.2** Let  $\mathcal{O}$  be an operad in  $k$ -DGM and  $A$  be an  $\mathcal{O}$ -algebra. We denote by  $\underline{A}$  the simplicial  $\mathcal{O}$ -algebra

$$\begin{aligned} \underline{A}_n &= A \amalg^{n+1}, \quad n \geq 0, \quad d_i : A \amalg^{n+1} \rightarrow A \amalg^n, \quad s_i : A \amalg^n \rightarrow A \amalg^{n+1} \\ d_i &= \begin{cases} id \amalg^i \amalg \nabla \amalg id \amalg^{n-i} & \text{if } i = 0, 1, \dots, n-1 \\ (\nabla \amalg id \amalg^n) \circ \tau_n & \text{if } i = n \end{cases}, \\ s_i &= id \amalg^i \amalg l \amalg id \amalg^{n-i}. \end{aligned}$$

The normalization  $N(\underline{A})$ , (see 2.2), of the simplicial  $\mathcal{O}$ -algebra  $\underline{A}$  is an  $\mathcal{O}$ -algebra, denoted  $C_*^{\mathfrak{H}}(\mathcal{O}, A)$ , and called the *Hochschild chain complex of the  $\mathcal{O}$ -algebra  $A$* . The homology of the  $C_*^{\mathfrak{H}}(\mathcal{O}, A)$  is the *operadic Hochschild homology*, denoted  $H_*^{\mathfrak{H}}(\mathcal{O}, A)$ .

**3.3** If  $(A, d_A)$  is any *associative* differential graded algebra supposed unital and augmented we denote by  $\overline{A}$  the kernel of the augmentation. The (classical) Hochschild chain complex is defined as follows:

$$\mathfrak{C}_* A = \{\mathfrak{C}_k A\}_{k \geq 0}, \quad \mathfrak{C}_k A = A \otimes s\overline{A}^{\otimes k},$$

with  $A = k \oplus \bar{A}$ . A generator of  $\mathfrak{C}_k A$  is written  $a_0[sa_1|sa_2|\dots|sa_k]$  if  $k > 0$  and  $a[ ]$  if  $k = 0$ . We set  $\epsilon_i = |a_0| + |a_1| + |sa_2| + \dots + |sa_{i-1}|$ ,  $i \geq 1$ . The differential  $d = d^1 + d^2$  is defined by:

$$\begin{aligned} d^1 a_0[a_1|a_2|\dots|a_k] &= da_0[a_1|a_2|\dots|a_k] - \sum_{i=1}^k (-1)^{\epsilon_i} a_0[a_1|\dots|da_i|\dots|a_k] \\ d^2 a_0[a_1|a_2|\dots|a_k] &= (-1)^{|a_0|} a_0 a_1[a_2|\dots|a_k] + \sum_{i=2}^k (-1)^{\epsilon_i} a_0[a_1|\dots| \\ &\quad |a_{i-1}a_i|\dots|a_k - (-1)^{|sa_k| \epsilon_k} a_k a_0[a_1|\dots|a_{k-1}] \end{aligned}$$

Consider the shuffle map, [18] 4.2.1,  $sh : \mathfrak{C}_* A \otimes \mathfrak{C}_* A \longrightarrow \mathfrak{C}_*(A \otimes A)$  defined by:

$$\begin{aligned} sh(a_0[a_1|a_2|\dots|a_n], b_0[b_1|b_2|\dots|b_m]) &= \\ (-1)^t \sum_{\sigma \in \Sigma_{n,m}} &= (-1)^{\epsilon(\sigma)} a_0 \otimes b_0[c_{\sigma(1)}|\dots|c_{\sigma(m+n)}] \end{aligned}$$

where  $t = |b_0|(|a_0| + \dots + |a_n|)$ ,  $c_{\sigma(i)} = \begin{cases} a_{\sigma(i)} \otimes 1, & 1 \leq i \leq n \\ 1 \otimes b_{\sigma(i-n)}, & n+1 \leq i \leq n+m \end{cases}$  and  $\epsilon(\sigma) = \sum |c_{\sigma(i)}||c_{\sigma(m+j)}|$ , summed over all pairs  $(i, m+j)$  such that  $\sigma(m+j) < \sigma(i)$ . Clearly,  $sh$  induces a chain map still denoted  $sh$ ,  $\mathfrak{C}_* A \otimes \mathfrak{C}_* A \longrightarrow \mathfrak{C}_*(A \otimes A)$ . Let  $HH_*(A)$  be the homology of  $\mathfrak{C}_* A$ .

If  $A$  is commutative (in the graded sense) then the multiplication  $\mu_A : A \otimes A \rightarrow A$  is a homomorphism of differential graded algebras. Thus the composite  $\mathfrak{C}_* \mu_A \circ sh : \mathfrak{C}_* A \otimes \mathfrak{C}_* A \rightarrow \mathfrak{C}_* A$  defines a multiplication on  $\mathfrak{C}_* A$  which makes it into a commutative differential graded algebra [18] 4.2.2.

If  $A$  is an associative (non commutative) differential graded algebra, there is no interesting product on  $\mathfrak{C}_*(A)$  while  $C_*^{\mathfrak{H}}(A, A)$  is naturally an associative differential graded algebra. Obviously,  $\mathfrak{C}_*(A)_{\#} \neq C_*^{\mathfrak{H}}(A, A)_{\#}$ . N. Bitjong and second named author, [3], have proved that for any strongly homotopy commutative  $k$ -algebra  $A$ , in the sense of [22], there is a well defined cup product on  $HH_*(A)$ , ([3]-Theorem 1), which is induced from a non canonical product on  $\mathfrak{C}_*(A)$ . In the formalism of operads, a strongly homotopy commutative  $k$ -algebra  $A$  is an associative  $\mathcal{B}_{\infty}$ -algebra, in the sense of [10]- 5.2, (see [3]-Proposition 2). The graded vector space  $C_*^{\mathfrak{H}}(\mathcal{B}_{\infty}, A)$  is not isomorphic to  $\mathfrak{C}_*(A)$ . An interesting question is: *Let  $A$  be a strongly homotopy commutative algebra  $A$  does  $H_*^{\mathfrak{H}}(\mathcal{B}_{\infty}, A) \cong HH_*(A)$  as commutative graded algebras?*

**3.4** Let  $A$  be an associative (unital) differential graded algebra. There is classically associated to  $A$  an other simplicial differential graded algebra,

which we denote  $\underline{A}$  and which is defined as follows (see [14]-Exemple 1.4):

$$\begin{aligned} \underline{A}_n &= A^{\otimes n+1}, n \geq 0, \quad d_i : A^{\otimes n+1} \rightarrow A^{\otimes n}, \quad s_i : A^{\otimes n} \rightarrow A^{\otimes n+1} \\ d_i &= \begin{cases} id^{\otimes i} \otimes \mu_A \otimes id^{\otimes n-i} & \text{if } i = 0, 1, \dots, n-1 \\ (\otimes \mu_A \otimes id^{\otimes n}) \circ \tau_n & \text{if } i = n \end{cases}, \\ s_i &= id^{\otimes i} \otimes 1 \otimes id^{\otimes n-i}. \end{aligned}$$

where  $\mu_A$  demotes the multiplication of  $A$  and  $\tau_n$  the map  $a_0 \otimes a_1 \otimes \dots \otimes a_n \mapsto (1)^{(|a_0|+\dots+|a_{n-1}|)|a_n|} \rightarrow a_n \otimes a_0 \otimes \dots \otimes a_{n-1}$ .

The complex  $\tilde{\mathfrak{C}}(A) = N_* \underline{A}$  is the *unreduced Hochschild chain complex*. By [19]-Chapter X-Corollary 2.2 and Theorem 9.1, the maps  $(id \otimes s^{\otimes n})$  define a quasi-isomorphism

$$(1) \quad \tilde{\mathfrak{C}}(A) \rightarrow \mathfrak{C}_*(A).$$

If we assume that  $A$  is commutative,  $\tilde{\mathfrak{C}}_*(A)$  is also a differential graded algebra. The multiplication is the shuffle product defined in the same way that the shuffle product on  $\mathfrak{C}_*(A)$ . Therefore the quasi-isomorphism (1) is a homomorphism of differential graded algebras.

**3.5 Theorem A.** *Assume that  $A$  is associative commutative differential graded algebra. There exists a natural quasi-isomorphism of commutative differential graded algebras*

$$C_*^{\mathfrak{H}}(\mathcal{C}, A) \rightarrow \mathfrak{C}(A).$$

**Remark.** If the  $\mathcal{C}$ -algebra  $A$  is considered as a  $\mathcal{C}_\infty$ -algebra then, by naturality there is a surjective homomorphism of  $\mathcal{C}_\infty$ -algebras

$$C_*^{\mathfrak{H}}(\mathcal{C}_\infty, A) \rightarrow C_*^{\mathfrak{H}}(\mathcal{C}, A).$$

Does this map induces an isomorphism in homology?

**Proof.** Let  $B$  be a  $\mathcal{C}$ -algebra. By universal property, there exists a natural isomorphism  $\Phi_{A,B} : A \amalg B \rightarrow A \otimes B$  of commutative differential graded algebras such that the following diagrams commute where we put  $\Phi_{A,A} = \Phi_1$ .

$$\begin{array}{ccc} A \amalg A & \xrightarrow{\nabla} & A & & A \amalg A & \xrightarrow{T} & A \amalg A \\ \Phi_1 \downarrow & & \parallel & , & \Phi_1 \downarrow & & \parallel \\ A \otimes A & \xrightarrow{\mu} & A & & A \otimes A & \xrightarrow{T} & A \otimes A \end{array}$$

where,  $\mu$  denotes the usual product on  $A \otimes A$  and  $T$  the usual twisting map. The associativity properties permit iteration so that we obtain for any  $n \geq 0$  an isomorphism  $\Phi_n : A \amalg^{n_1} \rightarrow A^{\otimes n+1}$ . These  $\Phi_n$ 's induce an isomorphism of simplicial differential graded modules  $\underline{A} \rightarrow \underline{\underline{A}}$  which in turn induces an isomorphism

$$(2) \quad C_*^{\mathfrak{S}}(A) = N_*(\underline{A}) \rightarrow N_*(\underline{\underline{A}}) = \tilde{\mathfrak{C}}_*(A).$$

Composition of the isomorphism (2) with the quasi-isomorphism (1) gives the quasi-isomorphism

$$(3) \quad C_*^{\mathfrak{S}}(A) \rightarrow \mathfrak{C}_*(A).$$

On the other hand, the  $\mathcal{C}$ -algebra structure on  $C_*^{\mathfrak{S}}(A)$  is such that the isomorphism (2) is an isomorphism of differential graded algebras. Thus (3) is a quasi-isomorphism of commutative differential graded algebras.  $\square$

**3.6** An operad  $\mathcal{C}_\infty$  is a Hopf operad "up to homotopy" (for the notion of Hopf operad we refer to [10]-5.3), that is to say it has a diagonal which is not coassociative but only up to homotopy. In this case, the tensor product of two  $\mathcal{C}_\infty$ -algebras  $A$  and  $B$  is a  $\mathcal{C}_\infty$ -algebra [7] with underlying differential graded module being the tensor product of the underlying differential graded modules, denoted  $A \otimes B$ . Indeed, Hinich [13] has proved that if the  $\mathcal{C}_\infty$ -algebras,  $A$  and  $B$  are cofibrant there exists a natural quasi-isomorphism

$$\Phi_{A,B} : A \amalg B \rightarrow A \otimes B.$$

Let  $f : A \rightarrow A'$  and  $g : B \rightarrow B'$  be two homomorphisms of  $\mathcal{C}_\infty$ -algebras. Then, by naturality of  $\Phi_{A,B}$ , we obtain the commutative diagram

$$\begin{array}{ccc} A \amalg B & \xrightarrow{f \amalg g} & A' \amalg B' \\ \Phi_{A,B} \downarrow & & \downarrow \Phi_{A',B'} \\ A \otimes B & \xrightarrow{f \otimes g} & A' \otimes B' \end{array}$$

If we assume that  $A$  and  $A'$  are cofibrant and that  $f$  and  $g$  are quasi-isomorphisms, then by [20]-Theorem 3.2, the homomorphism  $f \amalg g$  is a quasi-isomorphism. Therefore,  $f \otimes g$  is also a quasi-isomorphism.

**3.7** Assume that  $A = F(\mathcal{O}, V)$  is an almost free  $\mathcal{O}$ -algebra. Thus, we have the sequence of direct summands

$$\underline{A}_n \supset \mathcal{O}(k) \otimes_{\Sigma_k} (V^{\oplus n+1})^{\otimes k} \supset \mathcal{O}(k) \otimes_{\Sigma_k} (V^{\otimes k_0} \otimes V^{\otimes k_1} \otimes \dots \otimes V^{\otimes k_n}),$$

with  $k = k_0 + k_1 + \dots + k_n$ . Therefore, an element of  $\underline{A}_n$  is finite sum of elements of the form  $x \otimes v_1 \dots v_{k_0} \otimes v_{k_0+1} \dots v_{k_0+k_1} \otimes \dots \otimes v_{k_0+\dots+k_{n-1}+1} \dots v_k$  with  $x \in \mathcal{O}(k)$  and  $v_{k_0+\dots+k_{i-1}+1} \dots v_{k_0+\dots+k_i} \in V^{\otimes k_i}$  with usual convention  $V^{\otimes 0} = \mathbb{k}$ . With this notation we obtain explicit formulas for the map  $r$ ,  $l$  and  $\nabla$  defined in 3.1:

$$\begin{aligned} r : \underline{A}_0 &\rightarrow \underline{A}_1 & r(x \otimes v_1 \dots v_{k_0}) &= x \otimes v_1 \dots v_{k_0} \otimes 1 \\ l : \underline{A}_0 &\rightarrow \underline{A}_1 & l(x \otimes v_1 \dots v_{k_0}) &= x \otimes 1 \otimes v_1 \dots v_{k_0} \\ \nabla : \underline{A}_1 &\rightarrow \underline{A}_0 & \nabla(x \otimes v_1 \dots v_{k_0} \otimes v_{k_0+1} \dots v_{k_0+k_1}) & \\ & & &= x \otimes v_1 \dots v_{k_0} v_{k_0+1} \dots v_{k_0+k_1} . \end{aligned}$$

Therefore,

$$\begin{aligned} d_i : \underline{A}_{n+1} &\rightarrow \underline{A}_n , \\ d_i (x \otimes v_1 \dots v_{k_0} \otimes v_{k_0+1} \dots v_{k_0+k_1} \otimes \dots \otimes v_{k_0+\dots+k_{n-1}+1} \dots v_k) &= \\ \begin{cases} x \otimes v_1 \dots v_{k_0} \otimes \dots \otimes v_{k_0+k_i+1} \dots v_{k_0+\dots+k_{i+2}} \otimes \dots \\ \quad \otimes v_{k_0+\dots+k_{n-1}+1} \dots v_k & \text{if } i = 0, 1, \dots, n-1 \\ = x \otimes v_{k_0+\dots+k_n+1} \dots v_k v_1 \dots v_{k_0} \otimes \dots \otimes v_{k_0+\dots+k_{n-2}+1} \dots \\ \quad v_{k_0+\dots+k_{n-1}} & \text{if } i = n \end{cases} \\ s_i : \underline{A}_n &\rightarrow \underline{A}_{n+1} , \\ s_i (x \otimes v_1 \dots v_{k_0} \otimes v_{k_0+1} \dots v_{k_0+k_1} \otimes \dots \otimes v_{k_0+\dots+k_{n-1}+1} \dots v_k) &= \\ x \otimes v_1 \dots v_{k_0} \otimes \dots \otimes 1 \otimes \dots \otimes v_{k_0+\dots+k_i+1} \dots v_{k_0+\dots+k_{i+1}} \otimes \dots \\ \dots \otimes v_{k_0+\dots+k_{n-1}+1} \dots v_k . \end{aligned}$$

Let us denote by  $\underline{A}_n^+$  the submodule of the  $\mathbb{k}$ -module  $\underline{A}_n$  generated by the elements of the form  $x \otimes v_1 \dots v_{k_0} \otimes v_{k_0+1} \dots v_{k_0+k_1} \otimes \dots \otimes v_{k_0+\dots+k_{n-1}+1} \dots v_k$  with  $x \in \mathcal{O}(k)$  and  $v_{k_0+\dots+k_{i-1}+1} \dots v_{k_0+\dots+k_i} \in V^{\otimes k_i}$  such that each  $k_i > 0$ . The above formulas for  $d_i$  and  $s_i$  show that

- a) the graded module  $\underline{A}_n^+$  is stable for the  $d_i$ 's but not stable for the  $s_i$ 's.
- b) the submodule  $D\underline{A}_n$  of  $\underline{A}_n$  generated by all degenerate elements ( $D\underline{A}_0 = 0$ ) is exactly the submodule generated by the elements  $x \otimes v_1 \dots v_{k_0} \otimes v_{k_0+1} \dots v_{k_0+k_1} \otimes \dots \otimes v_{k_0+\dots+k_{n-1}+1} \dots v_k$  such that at least one  $k_i = 0$ .

Since  $C_*^{\mathfrak{S}}(\mathcal{O}, A) = N(\underline{A}) = \text{Tot}(\underline{A})/D(\underline{A})$  and since  $d_A(\underline{A}_n^+) \subset \underline{A}_n^+$  we have proved the first part of the next result.

**Theorem B.** *Assume that  $A = F(\mathcal{O}, V)$  is an almost free  $\mathcal{O}$ -algebra. Then the restriction of the natural chain equivalence  $\text{Tot}\underline{A} \rightarrow N_*\underline{A}$  to the total complex,  $\text{Tot}(\underline{A}^+)$  is an isomorphism of differential graded modules*

$$\text{Tot}(\underline{A}^+) \rightarrow C_*^{\mathfrak{S}}(\mathcal{O}, A) .$$

Moreover, this homomorphism is an isomorphism of  $\mathcal{O}$ -algebras.

**End of proof.** Let us precise first that the evaluation products of the simplicial algebra  $\underline{A}$  are determined by the maps

$$\mathcal{O}(k)_q \otimes \underline{A}_{p_1, q_1}^+ \otimes \underline{A}_{p_2, q_2}^+ \otimes \cdots \otimes \underline{A}_{p_k, q_k}^+ \rightarrow \underline{A}_{p_1+p_2+\cdots+p_k, q+q_1+\cdots+q_k}^+$$

which are explicitey given by the shuffle products and the evaluation product of  $\mathcal{O}$  (see proof of lemma 2.2). This implies that the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}(n)_q \otimes \underline{A}_{p_1, q_1}^+ \otimes \cdots \otimes \underline{A}_{p_n, q_n}^+ & \hookrightarrow & \mathcal{O}(n)_q \otimes \underline{A}_{p_1, q_1} \otimes \cdots \otimes \underline{A}_{p_n, q_n} \\ \downarrow & & \downarrow \\ \underline{A}_{p_1+\cdots+p_n, q+q_1+\cdots+q_n}^+ & \hookrightarrow & \underline{A}_{p_1+\cdots+p_n, q+q_1+\cdots+q_n} \end{array}$$

Therefore, each  $\underline{A}_n^+$  is a sub  $\mathcal{O}$ -algebra of  $\underline{A}_n$ . □

It results from the formula above that  $d_0 = d_1 : \underline{A}_1 \rightarrow \underline{A}_0$  and that  $d_i \left( \bigoplus_{p>0, q \in \mathbb{Z}} \underline{A}_{p, q} \right) \subset \bigoplus_{p>0, q \in \mathbb{Z}} \underline{A}_{p, q}$ . Thus we obtain:

**Proposition.** *Assume that  $A = F(\mathcal{O}, V)$  is an almost free  $\mathcal{O}$ -algebra. Then we have the natural splitting of  $\mathcal{O}$ -algebras*

$$\text{Tot}(\underline{A}^+) = (A, 0) \oplus \left( \bigoplus_{p>0} \underline{A}_{p, q}^+, d \right).$$

#### 4. Free loop space.

**4.1** Write **Top** (resp. **Costop**) for the category of topological spaces (resp. of cosimplicial topological spaces). The geometric realization of a cosimplicial set is the covariant functor

$$\|\cdot\| : \mathbf{Costop} \rightarrow \mathbf{Top}, \quad \underline{Z} \mapsto \|\underline{Z}\| = \mathbf{Costop}(\underline{\Delta}, \underline{Z}) \subset \prod_{n \geq 0} \mathbf{Top}(\Delta^n, \underline{Z}(n)),$$

where  $\|\underline{Z}\|$  is equipped with the topology induced by this inclusion. Here  $\underline{\Delta}$  denotes the cosimplicial space defined by  $\underline{\Delta}(n) = \Delta^n$  with the usual coface and codegeneracy maps. If  $\underline{Z}$  is any cosimplicial topological space, then

$N^*\underline{Z}$  is a simplicial cochain complex with total complex  $\text{Tot}(N^*\underline{Z})$ :

$$\begin{aligned} (\text{Tot}_n(N^*\underline{Z})) &= \bigoplus_{p-q=n} N^q \underline{Z}(p), \\ Dx &= \sum_{i=1}^p (-1)^i C^*(d_i) + (-1)^p \delta x, \quad x \in N^*\underline{Z}(p). \end{aligned}$$

The  $d_i$  are the coface operators of  $\underline{Z}$  and  $\delta$  is the internal differential of  $N^*(\underline{Z}(p))$ . Recall that in general the natural map  $\text{Tot}(N^*\underline{Z}) \rightarrow N^*(\|\underline{Z}\|)$  is not a weak equivalence, [5]. It results from lemma 2.2 and 2.3-5 that the total complex  $\text{Tot}(N^*(\underline{Z}))$  is naturally a  $\mathcal{C}_\infty$ -algebra and  $\text{Tot}(N^*(\underline{Z})) \rightarrow N^*(\|\underline{Z}\|)$  a homomorphism of  $\mathcal{C}_\infty$ -algebras.

Hereafter we denote  $N^*(N^*(\underline{Z}))$  the normalization of  $\text{Tot}(N^*(\underline{Z}))$ .

**4.2** One of the interest for considering cosimplicial spaces is the following result, [5]-Proposition 5.1, (see also [23]-Corollary 1): *If  $\underline{L}$  is a simplicial set and  $T$  a topological space then the cosimplicial space  $T^{\underline{L}}$  is such that there is a homeomorphism:*

$$\|\mathbf{T}^{\underline{L}}\| := \mathbf{Costop}(\underline{\Delta}, T^{\underline{L}}) \cong \mathbf{Top}(\|\underline{L}\|, T) = T^{|\underline{L}|}.$$

In particular, if we consider the simplicial set  $K$  defined as follows:  $K(n) = \mathbb{Z}/(n+1)\mathbb{Z}$ , and, if  $\bar{k}^n$  denotes an element in  $\mathbb{Z}/n\mathbb{Z}$ , the face maps  $d_i : K(n) \rightarrow K(n-1)$  with  $0 \leq i \leq n-1$  and the degeneracy maps  $s_j : K(n) \rightarrow K(n+1)$  with  $0 \leq j \leq n$  are:

$$d_i \bar{k}^{n+1} = \begin{cases} \bar{k}^n & \text{if } k \leq i \\ \overline{k-1}^n & \text{if } k > i \end{cases} \quad s_j \bar{k}^{n+1} = \begin{cases} \bar{k}^{n+2} & \text{if } k \leq j \\ \overline{k+1}^{n+2} & \text{if } k > j. \end{cases}$$

and  $d_n \bar{k}^{n+1} = \bar{k}^n$ . The geometric realization of  $K$ , [6] (proposition 1.4),  $|K|$  is homeomorphic to the circle  $S^1$ . Therefore, the cosimplicial model,  $\underline{X}$ , for the free loop space, used by Jones, [14],

$$\begin{aligned} \underline{X}(n) &= \text{Map}(K(n), X) = \underbrace{X \times \dots \times X}_{(n+1)\text{-times}} \\ d_i(x_0, x_1, \dots, x_n) &= (x_0, x_1, \dots, x_i, x_i, \dots, x_n), 0 \leq i \leq n \\ d_{n+1}(x_0, x_1, \dots, x_n) &= (x_0, x_1, \dots, x_n, x_0) \\ s_j(x_0, x_1, \dots, x_n) &= (x_0, x_1, \dots, x_j, x_{j+2}, \dots, x_n), 0 \leq j \leq n. \end{aligned}$$

is such that  $\|\underline{X}\| \cong \mathbf{Top}(\|K\|, X) = X^{S^1}$ , ( $\cong$  means homeomorphism). From lemma 2.2, we deduce then:

**Proposition.** *If  $X$  is simply connected, the natural map  $\text{Tot}(N^*\underline{X}) \rightarrow N^*(X^{S^1})$  is a quasi-isomorphism of  $\mathcal{C}_\infty$ -algebras.*

**4.3 Theorem C.** *Let  $X$  be a simply connected space such that each  $H^i(X)$  is finitely generated. Given an almost free model of the space  $X$*

$$\varphi_X : M_X = (F(\mathcal{C}, V), d) \rightarrow N^*(X)$$

*there exists a natural quasi-isomorphism of  $\mathcal{C}_\infty$ -algebras*

$$C_*^{\mathfrak{S}}(\mathcal{C}_\infty, M_X) \longrightarrow N^*(N^*(\underline{X})) .$$

**Proof.** Let  $\varphi_X : M_X = (F(\mathcal{C}, V), d) \rightarrow N^*(X)$  (resp.  $\varphi_Y : M_Y = (F(\mathcal{C}, V), d) \rightarrow N^*(Y)$ ) be a almost free model for the space  $X$  (resp. for the space  $Y$ ). By universal property, we obtain the commutative diagram

$$\begin{array}{ccccc} M_X & \xrightarrow{l} & M_X & \amalg & M_Y & \xleftarrow{r} & M_Y \\ \sim \downarrow \varphi_X & & \psi_{X,Y} \downarrow & & \sim \downarrow \varphi_Y & & \\ N^*(X) & \xrightarrow{N^*(pr_X)} & N^*(X \times Y) & \xleftarrow{N^*(pr_Y)} & N^*(Y) & & \end{array}$$

where  $pr_X$  and  $pr_Y$  (resp.  $i_X$  and  $i_Y$ ) are the natural projections (resp. inclusions). We have also the following commutative diagrams

$$\begin{array}{ccccc} M_X \amalg M_X & \xrightarrow{\nabla} & M_X & & M_X \amalg M_X & \xrightarrow{T^{\amalg}} & M_X \amalg M_X \\ \psi_{X,X} \downarrow & & \sim \downarrow \varphi_X & & \psi_{X,X} \downarrow & & \downarrow \psi_{X,X} \\ N^*(X \times X) & \xrightarrow{N^*(\Delta_X)} & N^*(X) & & N^*(X \times X) & \xrightarrow{N^*(T_X)} & N^*(X) \end{array}$$

where  $\Delta_X$  is the diagonal and  $T_X$  the topological interchange map.

By [20]-Lemma 5.2, we know that if each  $H^i(X)$  is finitely generated then  $\psi_{X,X}$  is a weak equivalence of differential graded module. It is now easy to prove that iteration furnishes a quasi-isomorphism of simplicial  $\mathcal{C}_\infty$ -algebras  $\underline{M}_X \rightarrow N^*(\underline{X})$ . □

**Acknowledgements.** The authors would like to thank Benoit Fresse for pointing out a mistake in a preliminary version and Martin Markl for his careful reading and suggestions.

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