Ext and inverse limits

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Dedicated to the memory of Reinhold Baer

Ext was introduced by Baer in [4]. Since then, through the works of Cartan, Eilenberg, Mac Lane, and many successors, the Ext functor has become an indispensable part of modern algebra.

It is well-known that the covariant Hom (= Ext^0) functor commutes with inverse limits. The Ext (= Ext^1) functor does not share this property: for example, any free group is an inverse limit of the divisible ones, [11]. Similarly, the contravariant Hom functor takes direct limits to the inverse ones. However, the corresponding property for $\mathrm{Ext}^1_R(-, M)$ holds if and only if M is pure-injective, [3].

More recently, the Ext functor has become an essential tool of approximation theory of modules, through the notion of a special approximation (see below for unexplained terminology). By Wakamatsu lemma, minimal approximations —envelopes and covers— are special. So in search of envelopes and covers of modules, one naturally deals with Ext, and Ext-orthogonal classes.

The key fact proved in [8] says that given a set $\mathcal{S}$ of modules there is always a special $\mathcal{S}^\perp$-preenvelope, μ_M, for any module M. Moreover, $\mathrm{Coker}\,\mu_M$ is $\mathcal{S}$-filtered, that is, $\mathrm{Coker}\,\mu_M$ is a particular well-ordered directed union (= direct limit of monomorphisms) such that the cokernels of all the successive embeddings are in $\mathcal{S}$. This fact is one of the main points in Enochs' proof of the Flat Cover Conjecture [6] as well as in recent works relating approximation theory, tilting theory, and the Finitistic Dimension Conjectures, [1], [2], [14], et al.

In view of these applications of the key fact of [8], it is natural to ask for its dualization, that is, for a possible construction of special ${}^\perp\mathcal{S}$-precovers whose kernels are $\mathcal{S}$-cofiltered. This paper provides for a construction of this sort.

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By the recent preprint of Eklof and Shelah [7], it is however not possible to obtain in ZFC a complete dual of the key fact of [8] for any module over any ring. Namely, [7] says that it is consistent with ZFC that the group of all rationals has no (special) Whitehead precover. So it is consistent that the dual result fails for $R = \mathbb{Z}$, $M = \mathbb{Q}$ and $\mathcal{S} = \{\mathbb{Z}\}$. This explains why our main result below —proved for any ring and any module— provides only for weak $\mathcal{S}$-precovers.

In fact, when defined appropriately, the required duals of the homological and category theoretic properties hold true. One can even overcome the non-exactness of the inverse limit functor. The reason for the failure of the general dual result is a set-theoretic fact: while the image of a small module mapped into a long well-ordered chain of modules will eventually sit in a member of the chain, that is, the map will factor through the given direct system of monomorphisms, there is no dual property ('slenderness') for well-ordered cochains of modules.

1 Preliminaries

Throughout this paper, R will denote a ring, M a (right R-) module and $\mathcal{C}$ a class of modules.

The notion of a $\mathcal{C}$-envelope generalizes the notion of an injective envelope going back to the pioneering work of Baer [5]:

A map $f \in \mathrm{Hom}_R(M, C)$ with $C \in \mathcal{C}$ is a **$\mathcal{C}$-preenvelope** of M provided the abelian group homomorphism $\mathrm{Hom}_R(f, C') : \mathrm{Hom}_R(C, C') \to \mathrm{Hom}_R(M, C')$ is surjective for each $C' \in \mathcal{C}$.

The $\mathcal{C}$-preenvelope f is a **$\mathcal{C}$-envelope** of M provided that $f = gf$ implies g is an automorphism for each $g \in \mathrm{End}_R(C)$.

Clearly, a $\mathcal{C}$-envelope of M is unique. In general, a module M may have many non-isomorphic $\mathcal{C}$-preenvelopes, but no $\mathcal{C}$-envelope. Nevertheless, if the $\mathcal{C}$-envelope of M exists, it is easily seen to be isomorphic to a direct summand in any $\mathcal{C}$-preenvelope of M provided that $\mathcal{C}$ is closed under direct summands and isomorphic copies.

Dually, $\mathcal{C}$-precovers and $\mathcal{C}$-covers are defined. These generalize the projective covers introduced by Bass in the 1960's.

The connection between Ext and approximations of modules is through the notion of a special approximation:

Let $\mathcal{C} \subseteq$ Mod-R. Define

$$\boldsymbol{\mathcal{C}}^{\perp} = \operatorname{Ker}\operatorname{Ext}^1_R(\mathcal{C},-) = \{N \in \text{Mod-}R \mid \operatorname{Ext}^1_R(C,N) = 0 \text{ for all } C \in \mathcal{C}\}$$
$$^{\perp}\boldsymbol{\mathcal{C}} = \operatorname{Ker}\operatorname{Ext}^1_R(-,\mathcal{C}) = \{N \in \text{Mod-}R \mid \operatorname{Ext}^1_R(N,C) = 0 \text{ for all } C \in \mathcal{C}\}.$$

For $\mathcal{C} = \{C\}$, we write for short $\boldsymbol{C}^{\perp}$ and $^{\perp}\boldsymbol{C}$ in place of $\{C\}^{\perp}$ and $^{\perp}\{C\}$, respectively.

Let $M \in$ Mod-R. A $\mathcal{C}$-preenvelope $f : M \to C$ of M is called **special** provided that f is injective and $\operatorname{Coker} f \in {}^{\perp}\mathcal{C}$. So a special $\mathcal{C}$-preenvelope may equivalently be viewed as an exact sequence

$$0 \longrightarrow M \xrightarrow{f} C \to D \longrightarrow 0$$

such that $C \in \mathcal{C}$ and $D \in {}^{\perp}\mathcal{C}$.

Dually, a $\mathcal{C}$-precover $f : C \to M$ of M is called **special** if f is surjective and $\operatorname{Ker} f \in \mathcal{C}^{\perp}$.

The next well-known result is the **Wakamatsu lemma**. It says that under rather weak assumptions on the class $\mathcal{C}$, $\mathcal{C}$-envelopes and $\mathcal{C}$-covers are special:

Lemma 1.1 *[10, §7.2] Let M be a module. Let $\mathcal{C}$ be a class of modules closed under extensions.*

1. *Assume $\mathcal{C}$ contains all injective modules. Let $f : M \to C$ be a $\mathcal{C}$-envelope of M. Then f is special.*

2. *Assume $\mathcal{C}$ contains all projective modules. Let $f : C \to M$ be a $\mathcal{C}$-cover of M. Then f is special.*

Another reason for investigating special approximations consists in a homological duality discovered by Salce:

Lemma 1.2 *[12] Let R be a ring and $\mathcal{A}$, $\mathcal{B}$ be classes of modules such that $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$. Then the following are equivalent:*

1. *Each module has a special $\mathcal{A}$-precover;*

2. *Each module has a special $\mathcal{B}$-preenvelope.*

The pairs of classes $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$ are called **cotorsion pairs**. The cotorsion pairs satisfying the equivalent conditions of Lemma 1.2 are called **complete**.

For any class of modules $\mathcal{C}$, both $({}^{\perp}(\mathcal{C}^{\perp}), \mathcal{C}^{\perp})$ and $({}^{\perp}\mathcal{C}, ({}^{\perp}\mathcal{C})^{\perp})$ are cotorsion pairs, called the cotorsion pairs **generated** and **cogenerated**, respectively, by the class $\mathcal{C}$. So there are many cotorsion pairs at hand. The following two theorems, due to Eklof and the author, say that there are also many *complete* cotorsion pairs. Before stating the results, we need more notation:

A sequence of modules $\mathcal{A} = (A_\alpha \mid \alpha \leq \mu)$ is a **continuous chain of modules** provided that $A_0 = 0$, $A_\alpha \subseteq A_{\alpha+1}$ for all $\alpha < \mu$ and $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ for all limit ordinals $\alpha \leq \mu$.

Let M be a module and $\mathcal{C}$ be a class of modules. Then M is **$\mathcal{C}$-filtered** provided that there are an ordinal κ and a continuous chain, $(M_\alpha \mid \alpha \leq \kappa)$, consisting of submodules of M such that $M = M_\kappa$, and each of the modules M_0, $M_{\alpha+1}/M_\alpha$ $(\alpha < \kappa)$ is isomorphic to an element of $\mathcal{C}$. The chain $(M_\alpha \mid \alpha \leq \kappa)$ is called a **$\mathcal{C}$-filtration** of M.

For example, if $\mathcal{C}$ is the set of all simple modules then the $\mathcal{C}$-filtered modules coincide with the semiartinian ones.

Theorem 1.3 *[8] Let $\mathcal{S}$ be a set of modules.*

1. *Let M be a module. Then there is a short exact sequence $0 \to M \hookrightarrow P \to N \to 0$ where $P \in \mathcal{S}^\perp$ and N is $\mathcal{S}$-filtered.*

   In particular, $M \hookrightarrow P$ is a special $\mathcal{S}^\perp$-preenvelope of M.

2. *The cotorsion pair generated by $\mathcal{S}$ is complete.*

Theorem 1.4 *[9] Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair cogenerated by a class of pure injective modules. Then $\mathfrak{C}$ is complete, and every module has a $\mathcal{B}$-envelope and an $\mathcal{A}$-cover.*

Taking the class of *all* pure injective modules in Theorem 1.4, we get that every module has a cotorsion envelope, and a flat cover, that is, the Flat Cover Conjecture holds true, cf. [6].

Suprisingly, Theorem 1.4 is not proven by a *dualization* of Theorem 1.3, but rather by its *application*, that is, by proving that $\mathfrak{C}$ is generated by a set of modules. The obvious question asking for a dualization of Theorem 1.3 is the main topic of the next section.

2 Cofiltrations and weak approximations

We start with fixing the notation for the dual setting:

Definition 2.1 1. Let μ be an ordinal and $\mathcal{A} = (A_\alpha \mid \alpha \leq \mu)$ be a sequence of modules. Let $(g_{\alpha\beta} \mid \alpha \leq \beta \leq \mu)$ be a sequence of epimorphisms (with $g_{\alpha\beta} \in \mathrm{Hom}_R(A_\beta, A_\alpha)$) such that $\mathcal{I} = \{(A_\alpha, g_{\alpha\beta}) \mid$

$\alpha \leq \beta \leq \mu\}$ is an inverse system of modules. $\mathcal{I}$ is called **continuous** provided that $A_0 = 0$ and $A_\alpha = \varprojlim_{\beta<\alpha} A_\beta$ for all limit ordinals $\alpha \leq \mu$.

Let $\mathcal{C}$ be a class of modules. Assume that the inverse system $\mathcal{I}$ is continuous. Then A_μ is called **$\mathcal{C}$-cofiltered** (by $\mathcal{I}$) provided that $\mathrm{Ker}(g_{\alpha,\alpha+1})$ is isomorphic to an element of $\mathcal{C}$ for all $\alpha < \mu$.

2. Similarly, we define **continuous inverse systems of exact sequences** for well-ordered inverse systems of short exact sequences of modules.

For example, if $R = \mathbb{Z}$ and $\mathcal{C} = \{\mathbb{Z}_p\}$ for a prime integer p then $\mathbb{J}_p$ is $\mathcal{C}$-cofiltered. Similarly, M^κ is $\{M\}$-cofiltered for any module M and any cardinal $\kappa \geq \omega$.

When trying to dualize Theorem 1.3, the first problem we face is the non-exactness of the inverse limit functor in general. Fortunately, in our particular setting, $\varprojlim$ is exact.

Lemma 2.2 *The functor $\varprojlim$ is exact on well-ordered continuous inverse systems of exact sequences.*

Proof. Let μ be a limit ordinal. Let $0 \longrightarrow C_\alpha \xrightarrow{h_\alpha} B_\alpha \xrightarrow{g_\alpha} A_\alpha \longrightarrow 0$ $(\alpha < \mu)$ be a continuous well-ordered inverse system of short exact sequences with connecting triples of epimorphisms $(x_{\beta\alpha}, y_{\beta\alpha}, z_{\beta\alpha})$ $(\beta \leq \alpha < \mu)$.

We will prove that the sequence $0 \longrightarrow C_\mu \xrightarrow{h_\mu} B_\mu \xrightarrow{g_\mu} A_\mu \longrightarrow 0$ is exact where $g_\mu = \varprojlim_{\alpha<\mu} g_\alpha$ and $h_\mu = \varprojlim_{\alpha<\mu} h_\alpha$.

Since $\varprojlim$ is always left exact, it suffices to prove that g_μ is surjective. Consider a sequence $a = (a_\alpha \mid \alpha < \mu) \in A_\mu \subseteq \prod_{\alpha<\mu} A_\alpha$. By induction on $\alpha < \mu$, we define a sequence $b = (b_\alpha \mid \alpha < \mu) \in B_\mu \subseteq \prod_{\alpha<\mu} B_\alpha$ such that $g_\mu(b) = a$.

Since g_0 is surjective, there exists $b_0 \in B_0$ such that $g_0(b_0) = a_0$.

If a is defined up to $\alpha < \mu$, we can take $u \in B_{\alpha+1}$ such that $g_{\alpha+1}(u) = a_{\alpha+1}$. Let $v = y_{\alpha,\alpha+1}(u)$. Then $g_\alpha(v) = z_{\alpha,\alpha+1}(a_{\alpha+1}) = a_\alpha$, so $b_\alpha - v \in \mathrm{Im}(h_\alpha)$. It follows that there exists $w \in C_{\alpha+1}$ such that $b_\alpha - v = y_{\alpha\alpha+1} h_{\alpha+1}(w)$.

Define $b_{\alpha+1} = u + h_{\alpha+1}(w)$. Then $y_{\alpha\alpha+1}(b_{\alpha+1}) = v + (b_\alpha - v) = b_\alpha$, and $g_{\alpha+1}(b_{\alpha+1}) = g_{\alpha+1}(u) = a_{\alpha+1}$.

For $\alpha < \mu$ limit, we put $b_\alpha = (b_\beta \mid \beta < \alpha) \in B_\alpha$. Since $g_\alpha = \varprojlim_{\beta<\alpha} g_\beta$, we get $g_\alpha(b_\alpha) = a_\alpha$ by induction premise. ■

Let $\mathcal{C}$ be a class of modules. Then $M \in {}^{\perp}\mathcal{C}$ whenever M is ${}^{\perp}\mathcal{C}$-filtered. This well-known homological fact has a dual, with well-ordered direct limits of monomorphisms replaced by well-ordered inverse limits of epimorphisms, cf. [8]:

Lemma 2.3 *Let $\mathcal{C}$ be a class of modules, and M be a $\mathcal{C}^{\perp}$-cofiltered module. Then $M \in \mathcal{C}^{\perp}$.*

Proof. W.l.o.g., we assume that $\mathcal{C} = \{N\}$ for a module N. Let $\mathcal{I} = \{(A_\alpha, g_{\alpha\beta}) \mid \alpha \leq \beta \leq \mu\}$ be a continuous inverse system of modules such that $M = A_\mu$ is $N^{\perp}$-cofiltered by $\mathcal{I}$. By induction on $\alpha \leq \mu$, we prove that $\mathrm{Ext}(N, A_\alpha) = 0$; the claim is just the case of $\alpha = \mu$.

Let $\alpha < \nu$. By assumption, the short exact sequence

$$0 \to K_\alpha \hookrightarrow A_{\alpha+1} \stackrel{g_{\alpha,\alpha+1}}{\to} A_\alpha \to 0$$

has $K_\alpha \in N^{\perp}$, so it induces the exact sequence

$$0 = \mathrm{Ext}(N, K_\alpha) \to \mathrm{Ext}(N, A_{\alpha+1}) \to \mathrm{Ext}(N, A_\alpha) = 0$$

with the middle term zero.

Suppose that α is a limit ordinal, so $A_\alpha = \varprojlim_{\beta<\alpha} A_\beta$. For each $\beta < \alpha$, denote by π_β the projection of A_α to A_β. Since all the inverse system maps are surjective, so is π_β.

Let $N \cong F/K$ where F is a free module. Denote by ϵ the inclusion of K into F. It remains to show that any homomorphism $\varphi \in \mathrm{Hom}(K, A_\alpha)$ can be extended to some $\phi \in \mathrm{Hom}(F, A_\alpha)$ so that $\varphi = \phi\epsilon$.

Take $\varphi \in \mathrm{Hom}(K, A_\alpha)$. By induction on $\beta < \alpha$, define $h_\beta \in \mathrm{Hom}(F, A_\beta)$ such that $h_\beta\epsilon = \pi_\beta\varphi$ and $g_{\gamma\beta}h_\beta = h_\gamma$ for all $\gamma \leq \beta$. For $\beta = 0$, put $h_0 = 0$. If $\beta < \alpha$ is a limit ordinal, then h_β is defined as the inverse limit of $(h_\gamma \mid \gamma < \beta)$. Let $\beta < \alpha$. By induction premise, $\mathrm{Ext}(N, A_{\beta+1}) = 0$, so there exists $k_{\beta+1}$ such that $k_{\beta+1}\epsilon = \pi_{\beta+1}\varphi$. Put $\delta = h_\beta - g_{\beta,\beta+1}k_{\beta+1}$. Then $\delta\epsilon = 0$, so δ induces a homomorphism $\overline{\delta} \in \mathrm{Hom}(N, A_\beta)$. Since $\mathrm{Ext}(N, K_\beta) = 0$, there is $\Delta \in \mathrm{Hom}(F, A_{\beta+1})$ such that $\Delta\epsilon = 0$ and $\overline{\delta} = g_{\beta,\beta+1}\overline{\Delta}$, so $\delta = g_{\beta,\beta+1}\Delta$. Then $h_{\beta+1} = k_{\beta+1} + \Delta$ satisfies $h_{\beta+1}\epsilon = \pi_{\beta+1}\varphi$ and $g_{\beta,\beta+1}h_{\beta+1} = h_\beta$, hence $g_{\gamma\beta+1}h_{\beta+1} = h_\gamma$ for all $\gamma \leq \beta + 1$.

Finally, by the inverse limit property, there is $\phi \in \mathrm{Hom}(F, A_\alpha)$ such that $\pi_\beta\phi = h_\beta$ for all $\beta < \alpha$. Then $\pi_\beta\phi\epsilon = \pi_\beta\varphi$ for all $\beta < \alpha$, so $\phi\epsilon = \varphi$. ∎

Before considering the dual of Theorem 1.3, we will need to dualize two elementary constructions of continuous direct systems of modules. Since the dual constructions are certainly not elementary, we give more details below:

(I) Let $\mathcal{I} = \{(N_\alpha, f_{\alpha\beta}) \mid \alpha \leq \beta \leq \mu\}$ be a continuous inverse system of modules. Let $\mathcal{E} : 0 \longrightarrow N_\mu \xrightarrow{\nu} P \to M \longrightarrow 0$ be an exact sequence of modules. For each $\alpha < \mu$, consider the pushout of ν and $f_{\alpha\mu}$:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
N_\mu & \xrightarrow{f_{\alpha\mu}} & N_\alpha \\
\nu \downarrow & & \downarrow \\
P & \xrightarrow{g_{\alpha\mu}} & P_\alpha \\
\downarrow & & \downarrow \\
M & = & M \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Using the pushout property, we obtain a continuous inverse system of modules (except for the condition $P_0 = 0$), $\mathcal{J} = \{(P_\alpha, g_{\alpha\beta}) \mid \alpha \leq \beta \leq \mu\}$, such that $P = P_\mu$ and $P_0 \cong M$. Moreover, the exact sequences $0 \longrightarrow N_\alpha \to P_\alpha \to M \longrightarrow 0$ form an inverse system with the inverse limit $0 \longrightarrow N \xrightarrow{\nu} P \to M \longrightarrow 0$, and $\operatorname{Ker} f_{\alpha\alpha+1} \cong \operatorname{Ker} g_{\alpha\alpha+1}$ for all $\alpha < \mu$. The inverse system $\mathcal{J}$ will be called the inverse system **induced** by $\mathcal{I}$ and $\mathcal{E}$.

(II) Conversely, assume that $\mathcal{J} = \{(P_\alpha, g_{\alpha\beta}) \mid \alpha \leq \beta \leq \mu\}$ is a continuous inverse system of modules (except that we allow $P_0 \neq 0$). Let $P = P_\mu$, $M = P_0$, and define $N_\alpha = \operatorname{Ker} g_{\alpha 0}$ and $f_{\alpha\beta} = g_{\alpha\beta} \restriction N_\beta$ for all $\alpha \leq \beta \leq \mu$. Then the exact sequences $0 \longrightarrow N_\alpha \xrightarrow{\subseteq} P_\alpha \xrightarrow{g_{\alpha 0}} M \longrightarrow 0$ with the maps $(f_{\alpha\beta}, g_{\alpha\beta}, id_M)$ form a continuous inverse system of exact sequences with the inverse limit $0 \longrightarrow N_\mu \xrightarrow{\subseteq} P \to M \longrightarrow 0$. In particular, $\mathcal{I} = \{(N_\alpha, f_{\alpha\beta}) \mid \alpha \leq \beta \leq \mu\}$ is a continuous inverse system of modules, and $\operatorname{Ker} f_{\alpha\alpha+1} \cong \operatorname{Ker} g_{\alpha\alpha+1}$ for all $\alpha < \mu$. The continuous inverse system $\mathcal{I}$ will be called the inverse system **derived** from $\mathcal{J}$.

It is easy to check that an application of (II) to a system $\mathcal{J}$ yields a derived system $\mathcal{I}$ such that P_α is a pushout of $f_{\alpha\mu}$ and $N_\mu \hookrightarrow P$ for each

$\alpha < \mu$, so an application of (I) to $\mathcal{I}$ and the sequence $0 \longrightarrow N_\mu \xrightarrow{\subseteq} P \to M \longrightarrow 0$ induces the original system $\mathcal{J}$. Similarly, applying (I), and then (II), we get back a copy of the original system $\mathcal{I}$ and the exact sequence $0 \longrightarrow N_\mu \xrightarrow{\nu} P \to M \longrightarrow 0$.

The dualization will provide only for weak approximations of modules in the following sense:

Definition 2.4 Let R be a ring, μ be a limit ordinal, M be a module, and $\mathcal{S}$ be a set of modules. Put $X = \prod_{S \in \mathcal{S}} S$, and let $0 \longrightarrow X \to I \xrightarrow{\pi} J \longrightarrow 0$ be an exact sequence with I injective.

An epimorphism $f : P \to M$ is a **weak special $^\perp\mathcal{S}$-precover** of M (of length μ) provided that

(1) $\operatorname{Ker} f$ is $\mathcal{S}$-cofiltered by an inverse system $\mathcal{I}$ (indexed by ordinals $\leq \mu$), and

(2) for each $x \in \operatorname{Hom}_R(P, J)$ which factors through $\mathcal{J}$, there is $y \in \operatorname{Hom}_R(P, I)$ satisfying $\pi y = x$.

Here, $\mathcal{J} = \{(P_\alpha, g_{\alpha\beta}) \mid \alpha \leq \beta \leq \mu\}$ denotes the inverse system of modules (whose inverse limit is P) induced by $\mathcal{I}$ and $\mathcal{E} : 0 \longrightarrow \operatorname{Ker} f \to P \xrightarrow{f} M \longrightarrow 0$. The term "$x$ factors through $\mathcal{J}$" means that there is an ordinal $\alpha < \mu$ such that $\operatorname{Ker} x \supseteq \operatorname{Ker} g_{\alpha\mu}$ (that is, x factors through $g_{\alpha\mu}$).

Remark 2.5 Condition (1) implies $\operatorname{Ker} f \in {}^\perp(\mathcal{S}^\perp)$ by Lemma 2.3 as required in the definition of a special $^\perp\mathcal{S}$-precover. In particular, any homorphism $h : N \to M$ with $N \in {}^\perp\mathcal{S}$ factors through f.

On the other hand, condition (2) is weaker than $P \in {}^\perp\mathcal{S}$. Of course, f is a special $^\perp\mathcal{S}$-precover iff $P \in {}^\perp\mathcal{S}$ (the latter says that for *each* $x \in \operatorname{Hom}_R(P, J)$ there is $y \in \operatorname{Hom}_R(P, I)$ such that $\pi y = x$).

Theorem 2.6 *Let R be a ring, M be a module, and $\mathcal{S}$ be a set of modules. Put $X = \prod_{S \in \mathcal{S}} S$, and let $0 \longrightarrow X \to I \xrightarrow{\pi} J \longrightarrow 0$ be an exact sequence with I injective.*

Then for each cardinal δ there is a limit ordinal $\mu \geq \delta$ and a weak special $^\perp\mathcal{S}$-precover $f : P \to M$ of M of length μ.

Proof. W.l.o.g., $\delta \geq \aleph_0$. By induction on $\alpha \leq \delta$, we define a continuous well-ordered inverse system of modules, $\mathcal{J}_\delta = \{(P_\alpha, g_{\alpha\beta} \mid \alpha \leq \beta \leq \delta)\}$ as follows: First, $P_0 = M$ and $g_{00} = id_M$.

Let $\alpha < \delta$ and $\kappa = \operatorname{card}(\operatorname{Hom}_R(P_\alpha, J))$. Let π_α be the product of κ-many copies of π. Then $\operatorname{Ker}(\pi_\alpha) \cong X^\kappa$. In particular, $\operatorname{Ker}(\pi_\alpha)$ is $\mathcal{S}$-cofiltered. Let φ_α be the canonical morphism from P_α to J^κ. For each $h \in \operatorname{Hom}_R(P_\alpha, J)$, denote by $\rho_h \in \operatorname{Hom}_R(I^\kappa, I)$ and $\sigma_h \in \operatorname{Hom}_R(J^\kappa, J)$

the canonical projections. Then $h = \sigma_h \varphi_\alpha$ and $\sigma_h \pi_\alpha = \pi \rho_h$ for each $h \in \operatorname{Hom}_R(P_\alpha, J)$.

The pullback of π_α and φ_α

$$\begin{array}{ccc} P_{\alpha+1} & \xrightarrow{g_{\alpha,\alpha+1}} & P_\alpha \\ \psi_\alpha \big\downarrow & & \varphi_\alpha \big\downarrow \\ I^\kappa & \xrightarrow{\pi_\alpha} & J^\kappa \end{array}$$

defines $P_{\alpha+1}$, $g_{\alpha,\alpha+1}$ and ψ_α.

If $\alpha \leq \delta$ is a limit ordinal, we put $P_\alpha = \varprojlim_{\beta<\alpha} P_\beta$, and let $g_{\beta\alpha}$ be the projection $P_\alpha \to P_\beta$. This gives the construction of the system $\mathcal{J}_\delta$. Put $P = P_\delta$ and $f = g_{0\delta}$.

Consider $x \in \operatorname{Hom}_R(P, J)$ such that x factors through $\mathcal{J}_\delta$, that is, such that there are $\alpha < \delta$ and $z \in \operatorname{Hom}_R(P_\alpha, J)$ with $x = z g_{\alpha\delta}$. Altogether, we have

$$x = z g_{\alpha\delta} = \sigma_z \varphi_\alpha g_{\alpha\delta} = \sigma_z \varphi_\alpha g_{\alpha,\alpha+1} g_{\alpha+1,\delta} = \sigma_z \pi_\alpha \psi_\alpha g_{\alpha+1,\delta} = \pi y$$

where $g = \rho_z \psi_\alpha g_{\alpha+1,\delta}$. This proves that the system $\mathcal{J} = \mathcal{J}_\delta$ satisfies condition (2) in 2.4.

In order to make sure that condition (1) holds for the derived inverse system, we will refine the construction of $\mathcal{J}_\delta$; so we fix $\alpha \leq \delta$ and let $\kappa = \operatorname{card}(\operatorname{Hom}_R(P_\alpha, J))$.

Consider the canonical continuous inverse system of exact sequences for the direct product X^κ: $\mathcal{L} = \{(X_\beta, \pi_{\beta\gamma}) \mid \beta \leq \gamma \leq \lambda\}$ where $X_{\beta+1} = X_\beta \oplus S_\beta$ and $S_\beta \in \mathcal{S}$, for all $\beta \leq \lambda$, and $X_\lambda = X$. We apply construction (I) above to $\mathcal{L}$ and to the exact sequence $\mathcal{F} : 0 \longrightarrow K \xrightarrow{\nu} P_{\alpha+1} \xrightarrow{g_{\alpha\alpha+1}} P_\alpha \longrightarrow 0$, where $K = X^\kappa \cong \operatorname{Ker}(\pi_\alpha)$.

By construction (I), $\mathcal{F}$ is the inverse limit of the continuous inverse system of short exact sequences $0 \longrightarrow X_\beta \to Q_\beta \to P_\alpha \longrightarrow 0$ ($\beta < \lambda$) with triples of epimorphisms $(u_{\beta\gamma}, v_{\beta\gamma}, 1_{P_\alpha})$ ($\beta \leq \gamma < \lambda$) such that $\operatorname{Ker} u_{\beta,\beta+1} \cong \operatorname{Ker} v_{\beta,\beta+1} \cong S_\beta \in \mathcal{S}$ for all $\beta < \lambda$.

Now, refining the inverse system $\mathcal{J}_\delta = (P_\alpha, g_{\alpha\beta} \mid \alpha \leq \beta < \delta)$ (so that its length becomes $\mu \geq \delta$) using the modules Q_γ ($\gamma < \lambda$) for each $\alpha < \delta$, we can assume that $\operatorname{Ker}(g_{\alpha,\alpha+1}) \in \mathcal{S}$ for all $\alpha < \mu$.

Finally, applying the construction (II) above to $\mathcal{J} = \mathcal{J}_\mu$, we get the exact sequence $\mathcal{E} : 0 \longrightarrow N_\mu \xrightarrow{\subseteq} P \xrightarrow{f} M \longrightarrow 0$ where N_μ is the inverse limit of the derived inverse system of modules $\mathcal{I} = \{(N_\alpha, f_{\alpha\beta} \mid \alpha \leq \beta \leq \mu)\}$ and $\operatorname{Ker} f_{\alpha\alpha+1} \cong \operatorname{Ker} g_{\alpha\alpha+1} \in \mathcal{S}$ for all $\alpha < \mu$. The latter says that N_μ is $\mathcal{S}$-cofiltered, so the inverse system $\mathcal{I}$ satisfies condition (1) in 2.4.

Applying construction (I) to $\mathcal{I}$ and $\mathcal{E}$, we get back $\mathcal{J}$ by Remark 2.5. Since $\mathcal{J}$ is a refinement of the old $\mathcal{J}_\delta$, $\mathcal{J}$ satisfies condition (2) by the argument above. ■

Comparing Theorems 2.6 and 1.3, it is natural to ask whether there is always an ordinal μ —possibly a large one— such that $P \in {}^\perp\mathcal{S}$, that is, such that the weak special ${}^\perp\mathcal{S}$-precover f is actually a special ${}^\perp\mathcal{S}$-precover of M.

In the recent work [7], Eklof and Shelah prove that it is consistent with ZFC + GCH that the answer is negative (for $R = \mathbb{Z}$, $S = \{\mathbb{Z}\}$ and $M = \mathbb{Q}$):

Theorem 2.7 *[7, Theorem 0.4] It is consistent with ZFC + GCH that the group of all rational numbers does not have a Whitehead precover.*

In fact, in the Eklof-Shelah model, *any* transfinite procedure attempting to produce a special ${}^\perp\mathbb{Z}$-precover of $\mathbb{Q}$ using non-split extensions with kernels $\cong \mathbb{Z}$ in non-limit steps, and inverse limits of the continuous inverse systems of epimorphisms in the limit steps, will never stop.

However, in particular cases, Theorem 2.6 can be strengthened considerably to provide for special approximations (in ZFC):

Proposition 2.8 *[13]. Let R and S be rings. Let $A \in S$-Mod-R and $B \in$ Mod-R. Denote by λ the number of generators of the left S-module $\mathrm{Ext}^1_R(B, A)$. Assume that $\mathrm{Ext}^1_R(A^\lambda, A) = 0$. Then there is a module $C \in$ Mod-R such that*

1. $\mathrm{Ext}^1_R(C, A) = 0$ *and*
2. *there is an exact sequence* $0 \to A^\lambda \to C \to B \to 0$ *in* Mod-R.

The point of Proposition 2.8 is that if $\mathcal{S} = \{A\}$ satisfies $\mathrm{Ext}^1_R(A^\lambda, A) = 0$ for all λ (big enough), then the inverse limits considered above are just direct products. This makes it possible to argue more directly —using products of systems of short exact sequences rather than general inverse limits. For mor details, we refer to [13].

Corollary 2.9 *Let R be a ring and C be a module such that $\mathrm{Ext}^1_R(C^\kappa, C) = 0$ for all κ. Then each module has a special ${}^\perp C$-precover whose kernel is C-cofiltered.*

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