# Selmer Groups over $\mathbb{Z}_p^d$-extensions

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## Notation:

$A/K$: an abelian variety defined over a global field.

$L/K$: a $\mathbb{Z}_p^d$-extension, unramified outside a finite set of places of $K$.

$X_L$: the Pontryagin dual of $\mathrm{Sel}_{p^\infty}(A/L)$.

$\Gamma := \mathrm{Gal}(L/K)$, $\Lambda_\Gamma := \mathbb{Z}_p[[\Gamma]]$.

Question I: Is $X_L$ finitely generated over $\Lambda_\Gamma$?

We view $A$ as a sheaf of the flat topology of $K$ and let $\mathcal{A}[p^m]$ denote the kernel of the multiplication by $p^m$ on them.

On the other hand, we use $A[p^n]$, $n = 1, \ldots, \infty$, to denote the usual groups of torsion points.

For a finite extension $F$ of $K$, $\mathrm{Sel}_{p^m}(F)$ is defined to be the kernel of the composition

$$\mathrm{H}^1(F, \mathcal{A}[p^m]) \longrightarrow \mathrm{H}^1(F, A) \longrightarrow \bigoplus_v \mathrm{H}^1(F_v, A),$$

$$\mathrm{Sel}_{p^\infty}(A/F) = \varinjlim_m \mathrm{Sel}_{p^m}(A/F),$$

and

$$\mathrm{Sel}_{p^\infty}(A/L) = \varinjlim_F \mathrm{Sel}_{p^m}(A/F).$$

$S$: the set of all ramified places (for $L/K$).

A local necessary and sufficient condition:

**Theorem. 1** *The Iwasawa module $X_L$ is finitely generated over $\Lambda_\Gamma$ if and only if at each place $v \in S$, the local cohomology group $\mathrm{H}^1(\Gamma_v, A(L_v))$ is cofinitely generated over $\mathbb{Z}_p$.*

Fact: If $K$ is a number field, then this local condition always holds (Tate's local duality Theorem+Mattuck's theorem that the $p$-completion of $A(K_v)$ is finite over $\mathbb{Z}_p$).

Fact: If $K$ is a function field, then verification of the condition can be found in various cases (by Bandini, Longhi, Ochiai, Trihan,-,...).

Fact: If $K$ is a function field, then the $p$-completion of $A(K_v)$ is of (countably) infinite rank over $\mathbb{Z}_p$ (Voloch, "*Diaphantine approximation on abelian varieties in characteristic* $p$", American J. of Math. **117**(1995), 1089-1095).

If $A$ has good reduction at a given place $v$, let $\widehat{A}$ denote the associated formal group. Write

$$\mathrm{H}^1(K_v, \widehat{A}) = \mathrm{H}^1(\mathrm{Gal}(\bar{K}_v/K_v), \widehat{A}(\mathcal{O}_{\bar{K}_v})).$$

**Theorem. 2** *Suppose $char.(K) = p$ and $A$ has supersingular reduction at a place $v$ that is ramified over $L/K$. Then the natural maps*

$$\mathsf{H}^1(\Gamma_v, \widehat{A}(\mathcal{O}_{L_v})) \longrightarrow \mathsf{H}^1(K_v, \widehat{A}) \longrightarrow \mathsf{H}^1(K_v, A)_p$$

*are isomorphisms. In particular, the cohomology group $\mathsf{H}^1(\Gamma_v, A(L_v))_p$ is of (countably) infinite corank over $\mathbb{Z}_p$.*

The proof of

$$\mathsf{H}^1(\Gamma_v, \widehat{A}(\mathcal{O}_{L_v})) \xrightarrow{\sim} \mathsf{H}^1(K_v, \widehat{A})$$

is based on a modification of the theory of "deeply ramified extension" (Coates-Greenberg, "*Kummer theory for Abelian varieties over local fields*", Invent. math. **124**(1996), 129-174).

$$\mathsf{H}^1(\Gamma_v, \widehat{A}(\mathcal{O}_{L_v})) \xrightarrow{\sim} \mathsf{H}^1(K_v, \widehat{A}) \xrightarrow{\sim} \mathsf{H}^1(K_v, A)_p.$$

The exact sequence

$$0 \longrightarrow \widehat{A}(\mathcal{O}_{\bar{K}_v}) \longrightarrow A(\bar{K}_v) \longrightarrow \bar{A}(\bar{\mathbb{F}}_v) \longrightarrow 0$$

induces the long exact sequence

$$\begin{array}{cccc} A(K_v) \twoheadrightarrow & \bar{A}(\mathbb{F}_v) & & \mathsf{H}^1(\mathbb{F}_v, \bar{A})_p \\ & \downarrow & & \uparrow \\ & \mathsf{H}^1(K_v, \widehat{A}) & \longrightarrow & \mathsf{H}^1(K_v, A)_p \end{array}$$

and proves

$$\mathsf{H}^1(K_v, \widehat{A}) \xrightarrow{\sim} \mathsf{H}^1(K_v, A)_p.$$

Similarly, we also have

$$\mathsf{H}^1(\Gamma_v, \widehat{A}(\mathcal{O}_{L_v})) \xrightarrow{\sim} \mathsf{H}^1(\Gamma_v, A(L_v)).$$

The last statement follows from the theorem of Voloch as well as Tate's local duality theorem.

The proof of Theorem 1 is standard.

First, we use Nakayama's Lemma to show that "$X_L$ is finitely generated over $\Lambda_\Gamma$ if and only if $\mathrm{Sel}_{p^\infty}(A/L)^\Gamma$ is confinitely generated over $\mathbb{Z}_p$".

Then we use the Hochschild-Serre spectral sequence to show the restriction map

$$\mathrm{H}^1(K, \mathcal{A}[p^\infty]) \longrightarrow \mathrm{H}^1(L.\mathcal{A}[p^\infty])^\Gamma$$

has cofinitely generated (over $\mathbb{Z}_p$) kernel and cokernel.

In the function field case, due to Voloch's theorem, we would need to make sure if the cokernel of the localization map

$$\mathrm{H}^1(K, \mathcal{A}[p^\infty]) \longrightarrow \bigoplus_{v \in S} \mathrm{H}^1(K_v, A)_p$$

is confinitely generated over $\mathbb{Z}_p$.

Fortunately, the affirmative answer has been provided by González-Avilés and Tan ("*A generalization of the Cassels-Tate dual exact sequence*", Math. Res. Lett. **14** (2007), no.2, 295-302).

From now on, we assume that $K$ is a function field of characteristic $p$ and $A$ has good ordinary reduction at each place in $S$.

Then $X_L$ is known to be finitely generated over $\Lambda_\Gamma$ ("*A Generalized Mazur's Theorem and its applications*", to appear in Trans. AMS).

Let $\chi_\Gamma(X_L)$ denote the characteristic ideal.

Question II: Functional equation? The automorphism $\iota : \Gamma \longrightarrow \Gamma$, $\gamma \mapsto \gamma^{-1}$, induced the ring automorphism $\iota_* : \Lambda_\Gamma \longrightarrow \Lambda_\Gamma$. Do we have

$$\iota_*(\chi_\Gamma(X_L)) = \chi_\Gamma(X_L)?$$

Suppose $L'/K$ is an intermediate $\mathbb{Z}_p^{d-1}$ extension of $L/K$ with $\Gamma' = \mathrm{Gal}(L'/K)$. Let

$$R_{L/L'} : \Lambda_\Gamma \longrightarrow \Lambda_{\Gamma'}$$

be the ring homomorphism induce from the restriction of Galois action $\Gamma \longrightarrow \Gamma'$.

Question III: What is the relation between $R_{L/L'}(\chi_\Gamma(X_L))$ and $\chi_{\Gamma'}(X_{L'})$?

Write $\mathrm{Gal}(L/L') = \Psi$.

For each $v$, let $\mathbb{F}_v$ be the residue field and let $\Pi_v$ denote the group formed by the connected components of the closed fiber (over $\bar{\mathbb{F}}_v$) of the Néron model of $A/K_v$ and let $\pi_v$ denote the order of the Sylow $p$-subgroup of $\Pi_v^{\mathrm{Gal}(\bar{\mathbb{F}}_v/\mathbb{F}_v)}$.

Let $g = \dim A$. Suppose that $A$ has good ordinary reduction $\bar{A}$ at $v$. Then eigenvalues of the Frobenius endomorphism $\mathsf{F}_v : \bar{A} \longrightarrow \bar{A}$ over $\mathbb{F}_v$ are, counted with multiplicities,

$$\alpha_1, ..., \alpha_g, q_v/\alpha_1, ..., q_v/\alpha_g$$

where $\alpha_1, ..., \alpha_g$ are eigenvalues of the (twist) matrix $\mathbf{u}$ of the action on the Tate module of $\bar{A}[p^\infty]$ by the Frobenius substitution $\mathrm{Frob}_v$ that topologically generates $\mathrm{Gal}(\bar{\mathbb{F}}_v/\mathbb{F}_v)$.

Suppose $L'/K$ is unramified at $v$ with the Frobenius element $[v]_{L'/K} \in \Gamma'$. Then put

$$f_v = \prod_{i=1}^{g} ([v]_{L'/K} - \alpha_i) \times \prod_{i=1}^{g} ([v]_{L'/K} - \alpha_i^{-1}) \in \Lambda_{\Gamma'}.$$

**Theorem. 3** *We have*

$$R_{L/L'}(\chi_\Gamma(X_L)) = \prod_v \vartheta_v \cdot \chi_{\Gamma'}(X_{L'}).$$

*Here $v$ runs through all places of $K$, and if $v \notin S$, then*

$$\vartheta_v = \begin{cases} \pi_v \Lambda_{\Gamma'}, & \textit{if} \quad \Psi_v \neq 0; \\ \Lambda_{\Gamma'}, & \textit{otherwise}; \end{cases}$$

*if $v \in S$, then*

$$\vartheta_v = \begin{cases} f_v \Lambda_{\Gamma'}, & \textit{if } v \textit{ is unramified over } L'/K; \\ \Lambda_{\Gamma'}, & \textit{otherwise.} \end{cases}$$

The above theorem holds for the cases where $d \geq 2$.

If $d = 1$, then $L' = K$ and $\Gamma' = 0$. In this case, put $\Lambda_0 = \mathbb{Z}_p$, $p^\infty \mathbb{Z}_p = (0)$ and set

$$\chi_0(X_K) = p^n \mathbb{Z}_p \subset \Lambda_0,$$

if

$$X_K = \oplus_i \mathbb{Z}_p / p^{n_i} \mathbb{Z}_p, \;\; n_i \in \mathbb{Z} \cup \{\infty\}, \;\; n = \sum n_i.$$

**Theorem. 4**

$$\chi_0(X_K) = 0,$$

*if and only if*

$$R_{L/K}(\chi_\Gamma(X_L)) = 0.$$

The theorem holds for any $d$.

Suppose $\Omega \subset \Gamma$ is an open subgroup and

$$\omega_1, ..., \omega_n \in \Gamma$$

is a set of representative of the cosets of $\Gamma/\Omega$. Write

$$\Lambda_\Omega = \mathbb{Z}_p[[\Omega]] \subset \Lambda_\Gamma.$$

For an $\Theta \in \Lambda_\Gamma$, write

$$\Theta = \sum_{i=1}^{n} \omega_i \cdot \Theta_i, \;\; \Theta_i \in \Lambda_\Omega,$$

and set

$$\rho_{\Gamma/\Omega}(\Theta) = \prod_{\phi \in \widehat{\Gamma/\Omega}} (\sum_{i=1}^{n} \phi(\omega_i) \cdot \Theta_i).$$

Fact: If $\chi_\Gamma(X_L) = \Theta\Lambda_\Gamma$, then

$$\chi_\Omega(X_L) = \rho_{\Gamma/\Omega}(\Theta)\Lambda_\Omega.$$

For a continuous character

$$\phi : \Gamma \longrightarrow \mu_{p^\infty},$$

let

$$\phi_* : \Lambda_\Gamma \longrightarrow \bar{\mathbb{Q}}_p$$

be the induced ring homomorphism so that

$$\phi_*(\gamma) = \phi(\gamma), \;\; \gamma \in \Gamma.$$

**Corollary. 5** *Suppose $F/K$ is a finite intermediate extension of $L/K$ with* Gal$(L/F) = \Omega$. *Then* Sel$_{p^\infty}(A/F)$ *is infinity, if and only if*

$$\phi_*(\chi_\Gamma(X_L)) = 0, \;\; \textit{for some } \phi \in \widehat{\Gamma/\Omega}.$$

Let $\mathrm{Sel}_{p^\infty}(A/F)^0$ denote the $p$-divisible part of $\mathrm{Sel}_{p^\infty}(A/F)$, let

$$\mathrm{Sel}_{p^\infty}(A/L)^0 = \varinjlim_{F} \mathrm{Sel}_{p^\infty}(A/F)^0.$$

and let $X_L^0$ denote the Pontryagin dual of $\mathrm{Sel}_{p^\infty}(A/L)^0$.

Then $\chi_\Gamma(X_L^0)$ divides $\chi_\Gamma(X_L)$.

For $\gamma \in \Gamma \setminus \Gamma^p$, $\zeta \in \mu_{p^\infty}$, let

$$f_{\gamma,\zeta} = \prod_{\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\zeta)/\mathbb{Q}_p)} (\gamma - {}^\sigma\zeta) \in \Lambda_\Gamma.$$

Then $f_{\gamma,\zeta}$ is irreducible. The ideal generated by an element of this type is called a flat ideal.

**Theorem. 6** *An irreducible factor of $\chi_\Gamma(X_L)$ divides $\chi_\Gamma(X_L^0)$, if and only if it is a flat ideal.*

The proof of the theorem is based on the following theorem of Monsky ("*On $p$-adic power series*", Math. Ann. **255**(1981), 217-227).

Let $W = \text{Hom}_{cont}(\Gamma, \mu_{p^\infty})$.

$T \subset W$ is called a $\mathbb{Z}_p$-flat, if there exists $\{\epsilon_1, ..., \epsilon_k\}$ expandable to a basis of $\Gamma$ (over $\mathbb{Z}_p$) and $\zeta_1, ..., \zeta_k \in \mu_{p^\infty}$ so that

$$T = \{\phi \mid \phi(\epsilon_i) = \zeta_i, i = 1, ..., k\}.$$

**Theorem. 7** *If $\Theta \in \Lambda_\Gamma$, then*

$$\{\phi \in W \mid \phi_*(\Theta) = 0\}$$

*is a finite union of $\mathbb{Z}_p$-flat.*

# A brief sketch of the proof of Theorem 3:

A finitely generated $\Lambda_\Gamma$-module $M$ uniquely determines a $\Lambda_\Gamma$-module

$$M_0 = \bigoplus_{k=1}^{m} \Lambda_\Gamma/(\mu_k), \tag{1}$$

where each $\mu_k$ is a power of an irreducible element, as well as a pseudo-isomorphism

$$i : M_0 \longrightarrow M.$$

$i$ is actually injective, since $M_0$ contains no non-trivial pseudo-null submodule.

Denote by $N$ the co-kernel of $i$, and apply the multiplication by $\psi - 1$ to the exact sequence

$$0 \longrightarrow M_0 \stackrel{i}{\longrightarrow} M \longrightarrow N \longrightarrow 0$$

to get the exact sequence

$$\begin{array}{ccccc} M_0^{\Psi} & \hookrightarrow & M^{\Psi} & \longrightarrow & N^{\Psi} \\ & & & & \downarrow \\ N/(\psi-1) & \twoheadleftarrow & M/(\psi-1) & \longleftarrow & M_0/(\psi-1), \end{array}$$

**Lemma. 8** (Greenberg) *There exist a $Z_p$ submodule $\Gamma_0$ of $\Gamma$ mapped isomorphically onto $\Gamma'$ under the projection $\Gamma \longrightarrow \Gamma'$ such that $N$ is a finitely generated torsion module over $\Lambda_{\Gamma_0}$.*

From the exact sequence

$$0 \longrightarrow N^{\Psi} \longrightarrow N \xrightarrow{\psi-1} N \longrightarrow N/(\psi-1) \longrightarrow 0,$$

we see that $\chi_{\Gamma_0}(N^{\Psi}) = \chi_{\Gamma_0}(N/(\psi-1))$, and hence

$$\chi_{\Gamma'}(N^{\Psi}) = \chi_{\Gamma'}(N/(\psi-1).$$

Consequently,

$$\chi_{\Gamma'}(M/(\psi-1)) = R_{L/L'}(\chi_{\Gamma}(M)) \cdot \chi_{\Gamma'}(M^{\Psi}).$$

$$\chi_{\Gamma'}(X_L/(\psi-1)) = R_{L/L'}(\chi_\Gamma(X_L)) \cdot \chi_{\Gamma'}(X_L^\Psi).$$

Main points:

‡ If $X_0$, $X_1$ are the Pontryagin dual of the kernel and cokernel of

$$\mathsf{Sel}_{p^\infty}(A/L') \longrightarrow \mathsf{Sel}_{p^\infty}(A/L)^\Psi,$$

then $X_0$ is pseudo-null (over $\Lambda_{\Gamma'}$), while

$$X_1 \sim (\bigoplus_v \prod_{w|v} \mathsf{H}^1(\Psi_w, A(L_w)))\widehat{}.$$

For this, we need to show

$$\mathsf{H}^1(L', \mathcal{A}[p^\infty]) \longrightarrow \mathsf{H}^1(L, \mathcal{A}[p^\infty])^\Psi$$

has finite cokernel.

$$\chi_{\Gamma'}(X_L/(\psi-1)) = R_{L/L'}(\chi_{\Gamma}(X_L)) \cdot \chi_{\Gamma'}(X_L^{\Psi}).$$

‡ $X_L^{\Psi}$ is pseudo-null: Because, up to a finite group,

$$\mathrm{Sel}_{p^\infty}(A/L) = (\psi-1)\,\mathrm{Sel}_{p^\infty}(A/L).$$

‡

$$\theta_v = \chi_{\Gamma'}((\prod_{w|v} \mathrm{H}^1(\Psi_w, A(L_w)))\widehat{\ } ).$$