Arithmetic of Shimura curves and the Birch and Swinnerton-Dyer conjecture

Shou-Wu Zhang

Abstract

This is the note that the author gave at the Centre de Recerca Matemática (CRM), Bellaterra, Barcelona, Spain on October 19–24, 2009. The aim of this series of lectures is to give a comprehensive description of some recent work [1, 2, 3] of the author and his students on generalisations of the Gross-Zagier formula, Euler systems on Shimura curves, and rational points on elliptic curves.

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1 Modular curves and cusp forms

1.1 Classical Modular curves

A group Γ in $\mathrm{SL}_2(\mathbb{Q})$ is called a *congruent subgroup* of $\mathrm{SL}_2(\mathbb{Q})$, if for some $g \in \mathrm{GL}_2(\mathbb{Q})$, $g\Gamma g^{-1}$ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ including a *full congruence subgroup* $\Gamma(N)$ for some N as follows:

$$\Gamma(N) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \quad \gamma \equiv 1 \pmod N\}.$$

The modular curve for such a Γ is defined as

$$Y_\Gamma = \Gamma\backslash\mathscr{H}^+, \qquad \mathscr{H}^+ = \{z \in \mathbb{C}, \mathrm{Im} z > 0\}$$

The action of Γ on $\mathscr{H}$ is the usual Mobiüs transformation. Two elements in Γ have the same action if and only if their ratio is ± 1. Thus the action depends only on the image $\widetilde{\Gamma}$ in $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\pm 1$. The complex structure on Y_Γ is induced from that of $\mathscr{H}$ with some modification at finitely many fixed points modulo $\widetilde{\Gamma}$. The complex conjugation of Y_Γ is identical to

$$Y_\Gamma^- = \Gamma\backslash\mathscr{H}^-, \qquad \mathscr{H}^- = \{z \in \mathbb{C}, \quad \mathrm{Im} z < 0\}.$$

The curve Y_Γ is not projective. A compactification is given by adding cusps:

$$X_\Gamma = Y_\Gamma \coprod \mathrm{Cusp}_\Gamma, \qquad \mathrm{Cusp}_\Gamma = \Gamma\backslash\mathbb{P}^1(\mathbb{Q}).$$

The complex structure of X_Γ at a cusp can also be defined in a standard way.

There are some morphisms among modular curves:

1. For two congruence subgroups with inclusion $\Gamma_1 \subset \Gamma_2$, one has a finite morphism of curves
$$\pi_{\Gamma_1,\Gamma_2} : X_{\Gamma_1} \longrightarrow X_{\Gamma_2}.$$
Notice that this morphism is an isomorphism if and only if $\widetilde{\Gamma}_1 = \widetilde{\Gamma}_2$.

2. for a congruence subgroup Γ and an element $g \in \mathrm{GL}_2(\mathbb{Q})_+$, the elements with positive determinants, the natural action of $g^{-1} : \mathscr{H} \longrightarrow \mathscr{H}^\pm$ induces a morphism
$$R_g : X_\Gamma \longrightarrow X_{g^{-1}\Gamma g}.$$
The action R_g is trivial on all X_Γ if and only if $g \in \mathbb{Q}^\times$.

Thus all modular curves X_Γ form a projective limit system X with an action by $\mathrm{PGL}_2(\mathbb{Q})_+$. The curve X_Γ in many sense is regarded as the quotient X/Γ. This projective system is dominated as any two modular curves X_{Γ_1} and X_{Γ_2} is dominated by a third one $X_{\Gamma_1\cap\Gamma_2}$. Moreover any such modular curve is dominated by a principal modular curve $X_{\Gamma(N)}$.

Each modular curve can be canonically defined over algebraic number $\bar{\mathbb{Q}}$ such that the above projections and the action of $\mathrm{PGL}_2(\mathbb{Q})_+$ are all defined over $\bar{\mathbb{Q}}$. For example, the principal modular curve $X_{\Gamma(1)}$ is isomorphic to $\mathbb{P}^1$ by using j-function and thus defined over $\mathbb{Q}$. The modular curve $X_{\Gamma(N)}$ can be defined over $\mathbb{Q}(\zeta_N)$ by using factor that $X_{\Gamma(N)}$ parameterized elliptic curves E over $\mathbb{C}$ with an oriented basis P_1, P_2 of $E[N]$.

The conjugation of a modular curve under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is also a modular curve. Thus the set of connected components of the projective system X has an action by $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ which commutes with action of $\mathrm{PGL}_2(\mathbb{Q})_+$. For each modular curve X_Γ, let $\widetilde{X}_\Gamma$ denote the union of X_Γ and its Galois conjugates. Then we obtain a projective system of curves over $\mathbb{Q}$ with action by $\mathrm{PGL}_2(\mathbb{Q})_+$.

To describe this projective system, it is better to use adelic language.

1.2 Adelic modular curves

We write $\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n = \prod_p \mathbb{Z}_p$ denote the completion of $\mathbb{Z}$ by congruences. For any abelian group G, let $\widehat{G} = G \otimes \widehat{\mathbb{Z}}$. Then the adeles over $\mathbb{Q}$ is defined to be $\mathbb{Q}$-algebra $\mathbb{A} = \widehat{\mathbb{Q}} \times \mathbb{R}$. We view $\mathbb{R}$ and $\mathbb{Q}_p = \mathbb{Q} \otimes \mathbb{Z}_p$ as quotient algebra of $\mathbb{A}$. For any open and compact subgroup U of $\mathrm{GL}_2(\widehat{\mathbb{Q}})$, we define a curve

$$Y_U = \mathrm{GL}_2(\mathbb{Q})\backslash \mathscr{H}^{\pm} \times \mathrm{GL}_2(\widehat{\mathbb{Q}})/U$$

Here the action of $\mathrm{GL}_2(\mathbb{Q})$ is on both $\mathscr{H}^{\pm} = \mathbb{C} \setminus \mathbb{R}$ and $\mathrm{GL}_2(\widehat{\mathbb{Q}})$ in a obvious way, and the action of U on $\mathscr{H}^{\pm}$ is trivial. The curve X_U depends only on the image $\bar{U}$ of U in $\mathrm{GL}_2(\widehat{\mathbb{Q}})/\mathbb{Q}^\times$.

This curve is not connected. The set of connected component is given by

$$\pi_0(Y_U) = \mathrm{GL}_2(\mathbb{Q})\backslash\{\pm 1\} \times \mathrm{GL}_2(\widehat{\mathbb{Q}})/U \simeq \mathbb{Q}_+^\times\backslash\widehat{\mathbb{Q}}^\times/\det U \simeq \widehat{\mathbb{Z}}^\times/\det(U).$$

Here the second map is given by determinant map on $\mathrm{GL}_2(\widehat{\mathbb{Q}})$ which is bijection because of strong approximation of $\mathrm{SL}_2(\mathbb{Q})$: for any open compact subgroup U of $\mathrm{SL}_2(\widehat{\mathbb{Q}})$,

$$\mathrm{SL}_2(\widehat{\mathbb{Q}}) = \mathrm{SL}_2(\mathbb{Q}) \cdot U.$$

The third morphism is given by the decomposition

$$\widehat{\mathbb{Q}}^\times = \mathbb{Q}_+^\times \cdot \widehat{\mathbb{Z}}^\times.$$

In other words we have a decomposition

$$\mathrm{GL}_2(\widehat{\mathbb{Q}}) = \coprod_{a \in \widehat{\mathbb{Z}}^\times/\det(U)} \mathrm{GL}_2(\mathbb{Q})_+ \begin{pmatrix} a & \\ & 1 \end{pmatrix} U.$$

For each $a \in \widehat{\mathbb{Z}}^\times/\det(U)$, let $\mathscr{H}$ maps to X_U by sending $z \in \mathscr{H}$ to the point represented by $z \times \begin{pmatrix} a & \\ & 1 \end{pmatrix}$. Then we get an embedding

$$Y_{U,a} := \Gamma(U,a)\backslash\mathscr{H}, \qquad \Gamma(U,a) = \begin{pmatrix} a & \\ & 1 \end{pmatrix} U \begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix} \cap \mathrm{GL}_2(\mathbb{Q})_+.$$

Thus we have shown that each adelic modular curve Y_U is a finite union of classical modular curves.

Each modular curve X_U here can be canonically defined over $\mathbb{Q}$ using the following modular interpretation: the curve X_U parameterizes the set of equivalent pairs (E, κ) where E is an elliptic curve over $\mathbb{C}$ and ϕ is a U-class of isomorphisms

$$\kappa : \quad \widehat{\mathbb{Q}}^2 \simeq H_1(E, \widehat{\mathbb{Q}}).$$

Here two pairs (E_1, κ_1) and (E_2, κ_2) are equivalent if there is an isogeny ϕ such that

$$H_1(\phi) \circ \kappa_1 = \kappa_2.$$

If we replace $E/\mathbb{C}$ by any elliptic curves E/S for any $\mathbb{Q}$-scheme S, then we get a functor $\mathscr{F}_U$. This functor has a coarse modular space which is the canonical model of X_U.

From our description of modular problem, it is not difficult to show that the curve X_U is defined over $\mathbb{Q}$, that each connected component of X_U is defined over $\mathbb{Q}^{\mathrm{ab}} = \mathbb{Q}(\zeta_\infty)$, and the induced action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \simeq \widehat{\mathbb{Z}}^\times$ on $\pi_0(X_U) = \widehat{\mathbb{Z}}^\times / \det(U)$ is given by the natural multiplication in $\widehat{\mathbb{Z}}^\times$.

In term of classical language, we have just shown that every modular curve X_Γ is defined over $\mathbb{Q}^{\mathrm{ab}}$, and that two curve X_{Γ_1} and X_{Γ_2} are conjugate under $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) = \widehat{\mathbb{Z}}^\times$ if and only if their closures U_1 and U_2 in $\mathrm{GL}_2(\widehat{\mathbb{Q}})$ are conjugate under the action of $\mathrm{GL}_2(\widehat{\mathbb{Q}})$. If this is the case, then they are conjugate under an element of the form $\begin{pmatrix} a & \\ & 1 \end{pmatrix}$ in group side and σ_a in Galois side.

As U varies, the curves X_U form a projective system using projection with respect to the inclusion in the groups. This projective system has an action by $\mathrm{GL}_2(\widehat{\mathbb{Q}})/\mathbb{Q}^\times$. In comparison with projective system introduced in the last section by sending X_Γ to its union X_U of Galois conjugates, the action of $\mathrm{GL}_2(\mathbb{Q})_+$ is compatible with the action of $\mathrm{GL}_2(\widehat{\mathbb{Q}})$ with natural embedding.

1.3 Classical cusp forms

Let k be an even integer. A *cusp form of weight* k for a congruence subgroup Γ is a holomorphic function $f : \mathscr{H} \longrightarrow \mathbb{C}$ such that

$$f(\gamma z) = (cz + d)^k f(z), \qquad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

and that f vanishes at each cusp in the sense that it has a Fourier expansion with respect to a local parameter:

$$f = \sum_{n>0} a_n q^n.$$

Let $S_k(\Gamma)$ denote the space of all cusp forms for Γ for weight k. Modular forms can be consider as a section of the line bundle $\mathscr{L}^k$ on Y_Γ where $\mathscr{L}$ is the Hodge bundle defined as follows:

$$\mathscr{L}_\Gamma = \Gamma \backslash \mathscr{H} \times \mathbb{C}$$

where the action of $\mathrm{GL}_2(\mathbb{R})_+$ on $\mathscr{H} \times \mathbb{C}$ is given by

$$\gamma(z, t) = \left(\gamma z, \quad \frac{\sqrt{\det \gamma}}{cz + d} t\right), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The bundle extends to X_Γ by taking an obvious trivialization at each cusp so that the forms holomorphic at cusp correspond to holomorphic sections of the bundle. With this trivialization, we also have an isomorphism of line bundles

$$\mathscr{L}_\Gamma^2 \simeq \Omega^1_{X_\Gamma}(\text{Cusps}).$$

In this way a cusp form of weight k is simply an element in

$$S_k(\Gamma) = H^0(X_\Gamma, \mathscr{L}_\Gamma^{k-2} \otimes \Omega_\Gamma).$$

We will only consider the cusp form in the following discussion.

The assignment $\Gamma \longrightarrow S_k(\Gamma)$ with pull-back maps form a direct limit system. The direct limit S_k can be identified with functions on $\mathscr{H}$ which is *cuspidal of weight k for some* Γ. This is an infinite dimensional space with a natural action by $\mathrm{PGL}_2(\mathbb{Q})_+$. One main problem is to decompose this representation into irreducible ones.

In classical language, this is achieved by working on Hecke operators. More precisely, for each congruence subgroup Γ, we denote $\mathbb{H}_\Gamma := \mathbb{C}[\Gamma \backslash \mathrm{PGL}_2(\mathbb{Q})_+/\Gamma]$ the Hecke algebra formed by finite linear combinations of double cosets with certain convolution. Then $\mathbb{H}_\Gamma$ acts on $S_k(\Gamma)$. In this way, we can decompose this space into irreducible spaces under Hecke algebra. The action can be made completely geometric in terms of algebraic correspondences described as follows.

For any $g \in \mathrm{GL}_2(\mathbb{Q})_+$ and any congruence subgroup Γ, one has a morphism

$$X_{\Gamma \cap g^{-1}\Gamma g} \longrightarrow X_\Gamma \times X_\Gamma$$

defined by natural projection for the first component and the composition of projection

$$X_{\Gamma \cap g \Gamma g^{-1}} \longrightarrow X_{g\Gamma g^{-1}}$$

and the action g. The correspondence on X_Γ defined by divisor $X_{\Gamma \cap g^{-1}\Gamma g}$ is called the Hecke correspondence of defined by g. Notice that this correspondence depends only on the double coset $\Gamma g \Gamma$. We can also define the neoclassical cusp forms of weight k for compact open subgroup U of $\mathrm{GL}_2(\widehat{\mathbb{Q}})$ as a function on

$$f: \quad \mathscr{H} \times \mathrm{GL}_2(\widehat{\mathbb{Q}}) \longrightarrow \mathbb{C}$$

which is holomorphic on $\mathscr{H}^\pm$ and invariant under right translation of U and such that

$$f(\gamma z, \gamma g) = \det \gamma^{-k/2} (cz + d)^k f(z, g), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})_+$$

and f vanishes at cusps. Let $S_k(U)$ denote the space of all such cusp forms then if Y_U has decomposition as an union of Y_Γ then $S_k(U)$ is a direct sum of $S_k(\Gamma)$.

Let S_k denote the union of all such spaces which has an action by $\mathrm{GL}_2(\widehat{\mathbb{Q}})$ by right translation. A major result in automorphic form is that this space can be decomposed into a topological sum of irreducible representations of $\mathrm{GL}_2(\widehat{\mathbb{Q}})$. The first step is to decompose it according to the central character ω of $\mathbb{Q}^\times \backslash \widehat{\mathbb{Q}}^\times$:

$$S_k = \oplus S_k(\omega)$$

where $S_k(\omega)$ is the space of cusp forms with character ω when translate under the center $\widehat{\mathbb{Q}}^\times$. Then the second step is to decompose

$$S_k(\omega) = \oplus \pi^\infty$$

where π^∞ are irreducible and admissible representations of $\mathrm{GL}_2(\widehat{\mathbb{Q}})$ with central character ω. These representations are automorphic, i.e., can be embedded into the space of functions on $\mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A})$. We will describe these representations in the next subsection.

1.4 Adelic cusp forms

Let ω be a character of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$ and we consider the space $L^2(\omega)$ of functions f on $\mathrm{GL}_2(\mathbb{A})$ such that

1. $f(\gamma z g) = \omega(z) f(g)$, for all $\gamma \in \mathrm{GL}_2(\mathbb{Q})$ and $z \in \mathbb{A}^\times$;
2. for a fixed Haar measure on dg on $\mathrm{GL}_2(\mathbb{A})/\mathbb{A}^\times$,

$$\int_{\mathrm{GL}_2(\mathbb{Q})\mathbb{A}^\times \backslash \mathrm{GL}_2(\mathbb{A})} |f|^2(g) dg < \infty.$$

This space has an action of $\mathrm{GL}_2(\mathbb{A})$ by right translation.

A major problem in the theory of automorphic forms is to decompose this into irreducible representations. In fact, this space can be decomposed into a direct sum of three orthogonal spaces: the cusp forms, the direct sum of one dimensional representations, and the continuous spectrum formed by Eisenstein series:

$$L^2(\omega) = L^2_{cusp}(\omega) \oplus L^2_{cont}(\omega) \oplus \widehat{\oplus}_{\chi^2=\omega} \mathbb{C}\chi.$$

Here the last sum $\widehat{\omega}$ is a topological sum over characters χ of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$ which are also viewed as characters of $\mathrm{GL}_2(\mathbb{A})$ after composing with determinant map $\mathrm{GL}_2 \longrightarrow \mathrm{GL}_1$.

The space $L^2_{cusp}(\omega)$ consists of functions f in $L^2(\omega)$ such that for almost all g,

$$\int_{N(\mathbb{Q}) \backslash N(\mathbb{A})} f\,(ng)\, dn = 0$$

where dn is a Haar measure on the uniportant radical of matrix $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. To compare this with classical or neoclassical cusp forms, we want to make an embedding:

$$S_k(\omega) \longrightarrow L^2_{cusp}(\omega)$$

when ω is trivial on $\mathbb{R}^\times$ and thus can be viewed as a character on $\mathbb{Q}^\times\backslash\widehat{\mathbb{Q}}^\times$. For any $g = g_\infty g_f \in \mathrm{GL}_2(\mathbb{A})$, and $f \in S_k$, define

$$F(g) = \deg g_\infty^{k/2}(ci+d)^{-k} f(g_\infty i, g_f), \qquad g_\infty = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It is clear that $F(g)$ is invariant under the left translation by $\mathrm{GL}_2(\mathbb{Q})$. Under the right action of $\mathrm{SO}(2,\mathbb{R})$ it has character χ^k where χ is an isomorphism $\mathrm{SO}(2,\mathbb{R}) \longrightarrow U(1) \subset \mathbb{C}^\times$.

The space $L_{cont}(\omega)$ is topologically spanned by Eisenstein series

$$E(g,\phi) = \sum_{\gamma \in B(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q})} \phi(\gamma g)$$

for a continuous function ϕ on $\mathrm{GL}_2(\mathbb{A})$ such that

$$\phi(nzbg) = \omega(z)\phi(g), \qquad n \in N(\mathbb{A}), z \in \mathbb{A}^\times, b \in B(\mathbb{Q})$$

and that $|\phi|$ is compactly supported on $B(\mathbb{Q})\mathbb{A}^\times N(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})$.

The space $L^2_{cusp}(\omega)$ is not easy to described concretely but it is a topological sum of irreducible representations of $\mathrm{GL}_2(\mathbb{A})$. To do so, we consider the space of *admissible* space $L^\infty_{cusp}(\omega)^\infty$ consisting of functions which has the following properties:

1. f is invariant under an open compact subgroup U of $\mathrm{GL}_2(\widehat{\mathbb{Q}})$;
2. f is smooth in $\mathrm{GL}_2(\mathbb{R})$, thus it admits an action by Lie algebra gl_2 and its universal enveloping algebra $U(\mathrm{gl}_2)$.
3. f has finite type under the action of O_2 and the center $\mathscr{Z}$ of $U(\mathrm{gl}_2)$, i.e., the subspace of generated f under actions of O_2 and $\mathscr{Z}$ is finite dimensional.

The definition of $L^\infty_{cups}(\omega)$ is dense in $L^2_{cusp}(\omega)$ and admits an action by $\mathrm{GL}_2(\widehat{\mathbb{Q}})$ and $(\mathrm{gl}_2, \mathrm{O}_2)$ but not on $\mathrm{GL}_2(\mathbb{R})$. In fact, the action of a $g \in \mathrm{GL}_2(\mathbb{R})$ will change property 3 here to analogous one with O_2 replaces by $g\mathrm{O}(2)g^{-1}$. The space $L^\infty_{cusp}(\omega)$ is a direct (algebraic sum) of irreducible representations:

$$L^\infty_{cusp}(\omega) = \oplus_\pi \pi$$

where π run through irreducible representations of $(\mathrm{gl}_2, \mathrm{O}_2) \times \mathrm{GL}_2(\widehat{\mathbb{Q}})$ with central character ω.

Each irreducible representation of $(\mathrm{gl}_2, \mathrm{O}_2) \times \mathrm{GL}_2(\widehat{\mathbb{Q}})$ is a tensor product of irreducible representations:

$$\pi = \otimes_{p \le \infty} \pi_p$$

where π_∞ is an irreducible representation of $(\mathrm{gl}_2, \mathrm{O}_2)$ and π_p is an irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. The tensor product is defined with respect a base element $e_p \in \pi_p^{\mathrm{GL}_2(\mathbb{Z}_p)}$ for almost all p:

$$\pi = \varinjlim_S \otimes_{p \in S} \pi_p$$

where the map form the tensor product over S to the tensor product with a bigger T is given by tensoring with $\otimes_{p \in T\backslash S} e_p$ when S is sufficiently large.

The local representation π_p is also completely classified. When p is finite, it classified as three classes: the principal, special, or supercuspidal. The representation π_p is supercuspidal if and only if $\mathrm{Hom}_{N_p}(\pi_p, \mathbb{C}) = 0$ where N_p is the unipotent radical of the Boreal group. Otherwise, there is a character χ of the Borel group such that π is included into $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi)$. If the quotient is 0, then we called that π_p is principal. Otherwise it called a spacial representation; in this case it has an one dimensional quotient.

In archimedean case, we may first construct representations for $\mathrm{GL}_2(\mathbb{R})$ and then use process (2) and (3) to construct representation for $(\mathrm{gl}_2, \mathrm{O}_2)$. We know that every irreducible representation π_∞ is a subrepresentation of $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{R})}(\chi)$. If the quotient is trivial, then we get principal representation. Otherwise, we have a quotient isomorphic to the representation of $\mathrm{GL}_2(\mathbb{R})$ on the homogenus polynomial of degree $k-1$. We call this representation of discrete series of weight k and denoted it as D_k.

Let $\widetilde{S}_k$ denote admissinle vectors in the subspace of $\mathrm{GL}_2(\mathbb{A})$ generated by right translation of $\mathrm{GL}_2(\mathbb{R})$. Then we have

$$\widetilde{S} = D_k \otimes S_k = \oplus \pi$$

where D_k is the weight k representation of $\mathrm{GL}_2(\mathbb{R})$.

2 Jacobians and CM-points

2.1 Picard varieties

For each group U of $\mathrm{GL}_2(\widehat{\mathbb{Q}})$, let $\mathrm{Pic}(X_{U,\mathbb{Q}})$ denote the Picard group of line bundles on $X_{U,\bar{\mathbb{Q}}}$ and let $\mathrm{Pic}^0(X_{U,\bar{\mathbb{Q}}})$ denote the subgroup of line bundles with degree 0 on every connected component. Then we have an exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X_{U,\bar{\mathbb{Q}}}) \longrightarrow \mathrm{Pic}(X_{U,\bar{\mathbb{Q}}}) \longrightarrow \oplus_{\pi_0(X_U)}\mathbb{Z} \longrightarrow 0$$

where the last map is given by degree map. We tenor this exact sequence with $\mathbb{Q}$ then we have an canonical spliting of this exact sequence by the Hodge class $c_1(\mathscr{L}_\Gamma)$ in $\mathrm{Pic}(X_\Gamma)_\mathbb{Q}$. As Hodge class is compatible with pull-backs, taking inductive limit, we obtain

$$\mathrm{Pic}(X_{\bar{\mathbb{Q}}})_\mathbb{Q} = \mathrm{Pic}^0(X_{\bar{\mathbb{Q}}})_\mathbb{Q} \oplus \mathrm{Pic}^{Hodge}(X_{\bar{\mathbb{Q}}})_\mathbb{Q}$$

where $\mathrm{Pic}^{Hodge}(X)_\mathbb{Q}$ is the group of Hodge classes which is isomorphic to the space of locally constant function on $\pi^0(X_{\bar{\mathbb{Q}}})$. With respect to the action of $\mathrm{GL}_2(\widehat{\mathbb{Q}})$, this splitting is a splitting of representations of $\mathrm{GL}_2(\widehat{\mathbb{Q}})$. In the following we want to decompose these spaces into irreducible representations of $\mathrm{GL}_2(\widehat{\mathbb{Q}})$.

The decomposition of the Hoge part as it isomorphic to locally constant functions on $\mathbb{Q}_+^\times\backslash\widehat{\mathbb{Q}}^\times$. Thus it has an decomposition:

$$\mathrm{Pic}^{Hodge}(X_{\bar{\mathbb{Q}}})_\mathbb{Q} = \oplus_{\{\chi\}}\mathbb{Q}(\chi)$$

where the sum is over conjugacy classes of complex characters χ of $\mathbb{Q}_+^\times\backslash\widehat{\mathbb{A}}^\times$ and $\mathbb{Q}(\chi)$ is the extension of $\mathbb{Q}$ by adding values of χ.

To study action of $\mathrm{GL}_2(\widehat{\mathbb{Q}})$ on $\mathrm{Pic}^0(X_{\bar{\mathbb{Q}}})_{\mathbb{Q}}$, we use Jacobian. For each compact open compact subgroup U of $\mathrm{GL}_2(\widehat{\mathbb{Q}})$, let J_U denote the Jacobian variety of X_U. Then J_U is an abelian variety defined over $\mathbb{Q}$ such that

$$\mathrm{Pic}^0(X_{U,\bar{\mathbb{Q}}}) = J_U(\bar{\mathbb{Q}}).$$

Over $\mathbb{Q}^{\mathrm{ab}}$, if X_U is a union of X_Γ, then $J_U = \prod J_\Gamma$ is the product of the Jacobians of X_Γ. The pull-back morphism gives a direct limit system (J_U). The limit is denoted as J which admits an action by $\mathrm{GL}_2(\widehat{\mathbb{Q}})$. To understand this action, we look its action on cotangent space as a $\mathbb{Q}$-vector spaces:

$$\Omega^1_J = \lim_K \Omega^1_{J_K} = \lim_K \Omega^1_{X_K} = \Omega^1_X.$$

Then we have a decomposition into irreducible $\mathbb{Q}$-irreducible represenations under $\mathrm{GL}_2(\widehat{\mathbb{Q}})$:

$$\Omega^1_X = \oplus \Pi.$$

Notice that when tensoring with $\mathbb{C}$, $\Omega^1_{X_\mathbb{C}}$ is isomorphic to S_2, the space of cuspidal forms of weight 2, we have

$$\Omega^1_X \otimes \mathbb{C} = \oplus \pi^\infty$$

where the sum runs through the set of cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$ of weight 2. Thus the set of representations Π here are parameterized bijectively by the set of conjugacy classes of cuspidal representation by the relation

$$\Pi \otimes \mathbb{C} = \oplus \text{Galois conjugates of a } \pi^\infty$$

for some cuspidal representation of weight 2 of $\mathrm{GL}_2(\mathbb{A})$. Let $\mathbb{Q}(\Pi) = \mathrm{End}(\Pi)$ then $\mathbb{Q}(\Pi)$ is a number field and every π^∞ obtained by the base change with respect to some embedding $\mathbb{Q}(\Pi) \longrightarrow \mathbb{C}$. In this way, J has a decomposition

$$J \otimes \mathbb{Q} = \oplus_\Pi \Pi \otimes_{\mathbb{Q}(\Pi)} (A_\Pi \otimes \mathbb{Q})$$

where $A(\Pi)$ is an abelian variety with a multiplication by $\mathbb{Q}(\Pi)$ with dimension $[\mathbb{Q}(\Pi) : \mathbb{Q}]$.

To construct A_Π rigorously, one has to study the action of $\mathrm{GL}_2(\widehat{\mathbb{Q}})$ on $H^1(J, \mathbb{Q}) = H^1(X, \mathbb{Q})$ together with its Hodge structure. Let

$$V_\Pi = \mathrm{Hom}_{\mathrm{GL}_2(\widehat{\mathbb{Q}})}(\Pi, H^1(X, J))$$

Then V_Π is a two dimensional vector space over $\mathbb{Q}(\Pi)$ with a Hodge structure and we have a decomposition:

$$H^1(J, \mathbb{Q}) = \oplus_\Pi \Pi \otimes V_\Pi.$$

The Galois action $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $H^1(J, \widehat{\mathbb{Q}})$ also induces a Galois action on $\widehat{V}$. Pick up a appropriate lattice Λ_Π of V_Π so that $\widehat{\Lambda}$ is invariant under $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ to define abelian variety A_Π.

By work of Eichler, Shimura, Langlands, Caroyal, and Faltings, the abelian variety A_Π is characterized by the following property:

$$L(s, A_\Pi) = L(s-1/2, \Pi) := \prod_{\sigma:\mathbb{Q}(\Pi)\longrightarrow\mathbb{C}} L(s-1/2, \Pi\otimes_\sigma \mathbb{C}).$$

In summary, we have shown that

$$\mathrm{Pic}^0(X_{\bar{\mathbb{Q}}})_{\mathbb{Q}} = J(\bar{\mathbb{Q}})_{\mathbb{Q}} = \oplus_\Pi \Pi \otimes_{\mathbb{Q}(\Pi)} A_\Pi(\bar{\mathbb{Q}})_{\mathbb{Q}}.$$

2.2 Albanese

The Albanese variety $\mathrm{Alb}(X)$ of X is defined as the projective limit of $\mathrm{Alb}(X_U)$. In the following we want to embedding X into $\mathrm{Alb}(X_U)$. To do this we need to define the Hodge class ξ_Γ for each connected curve X_Γ:

$$\xi_\Gamma = c_1(\mathscr{L}_\Gamma)/\deg \mathscr{L}_\Gamma.$$

Concretely, ξ_Γ can be defined as follows:

$$\xi_\Gamma = \left\{ c_1(\Omega_{X_\Gamma}) + \sum_{x\in X_\Gamma}\left(1-\frac{1}{e_x}\right)x\right\} \Big/ \mathrm{vol}(X_\Gamma)$$

Here for each $x \in X_\Gamma$, e_x is the cardinality of the stabilizer of x in $\bar{\Gamma}$, and $\mathrm{vol}(X_\Gamma)$ is the volume of X_Γ computed with respect the measure induced from $dxdy/(2\pi y^2)$ on $\mathscr{H}$.

The classes ξ_Γ for a projective system for each connected component X formed by a projective system of X_Γ's. Thus we may make a map

$$X \longrightarrow \mathrm{Alb}(X)\otimes\mathbb{Q}, \qquad x \longrightarrow \text{class of } x - \xi_x$$

where ξ_ξ is the Hodge class in the connected component of X containing x.

On each curve J_U, there is a Neron–Tate height pairing

$$\langle\cdot,\cdot\rangle : \quad J_U(\bar{\mathbb{Q}})\otimes J_U(\bar{\mathbb{Q}}) \longrightarrow \mathbb{R}.$$

The projection formula gives a hermitian pairing

$$\langle\cdot,\cdot\rangle : \quad \mathrm{Jac}(X)(\bar{\mathbb{Q}})_{\mathbb{C}} \times \mathrm{Alb}(X)(\bar{\mathbb{Q}})_{\mathbb{C}} \longrightarrow \mathbb{R}.$$

2.3 CM-points

Let E be an imaginary quadratic extension of $\mathbb{Q}$. Then for each embedding of $\widehat{\mathbb{Q}}$-algebra $\widehat{E} \longrightarrow M_2(\widehat{\mathbb{Q}})$, we consider the subscheme $X^{E^\times}$ of X fixed by $E^\times$. The embedding $\widehat{E} \longrightarrow M_2(\widehat{\mathbb{Q}})$ is unique up conjugate, the subscheme $X^{E^\times}$ is unique up translation by an element in $\mathrm{GL}_2(\widehat{\mathbb{Q}})$.

The scheme $X^{E^\times}$ has an action by the normalizer H of $E^\times$ in $\mathrm{GL}_2(\widehat{\mathbb{Q}})/E^\times$. A simple calculation shows that H is generated by $\widehat{E}^\times$ and an element j such that $jx = \bar{x}j$ for all $x \in \widehat{E}^\times$, and that $X^{E^\times}(\bar{\mathbb{Q}})$ is a principal homogenous space of H. On other hand, the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ commutes with action of H. Thus the Galois action is given by a homomorphism

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow H.$$

The theory of complex multiplication says that every point in $X^{E^\times}$ is defined over E^{ab} and that the Galois action is given by the reciprocity map

$$\mathrm{Gal}(E^{\mathrm{ab}}/E) \simeq H.$$

Now let P be any point in $X^{E^\times}(E^{\mathrm{ab}})$ and let χ be any character of $\mathrm{Gal}(E^{\mathrm{ab}}/E)$. Then we define a class

$$P_\chi = \int_{\mathrm{Gal}(E^{\mathrm{ab}}/E)} \chi^{-1}(\sigma)(P^\sigma - \xi_{P^\sigma})d\sigma \in \mathrm{Alb}(X)(E^{\mathrm{ab}}) \otimes \mathbb{C}$$

where ξ_{P^σ} is the Hodge class in the component of X containing P^σ, and $d\sigma$ is a Haar measure on $\mathrm{Gal}(E^{\mathrm{ab}}/E)$. We will normalize it so that the total volume is $2L(2,\eta)$ with η is the quadratic extension of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ corresponding to the extension $E/\mathbb{Q}$.

2.4 Neron–Tate distrbution

For each $f \in C_0^\infty(\mathrm{GL}_2(\widehat{\mathbb{Q}})/\mathbb{Q}^\times, \mathbb{C})$ bi-invariant under an open compact subgroup U of $\mathrm{GL}_2(\widehat{\mathbb{Q}})/\mathbb{Q}^\times$, we can define a cycle

$$Z(f)_U = \sum_{g \in K\backslash \mathrm{GL}_2(\widehat{\mathbb{Q}})/U\mathbb{Q}^\times} f(g)Z(g)_U$$

where $Z(g)_U$ is the divisor on $X_U \times X_U$ defined as the image of two morphisms form $X_{U\cap gUg^{-1}}$ by the inclusion projection and the composition of the projection on to $X_{gUg^{-1}}$ and a right multiplication by g. The cycle $Z(f)_{U*}$ acts on cycles on X_U in the usual way which we denote as $T(f)$.

It is easy to see that the this definition is compatible with pull-back of cycles and this define en element in

$$Z(f) \in \mathrm{Ch}^1(X \times X)_{\mathbb{C}} := \lim_{\to U} \mathrm{Ch}^1(X_U \times X_U) \otimes \mathbb{C}.$$

From this property we see that the actions

$$T(f)_U : \quad \mathrm{Jac}(X_U)_{\mathbb{C}} \longrightarrow \mathrm{Jac}(X_U)$$

is compatible with pushward in the first variable and pull-back in the second variable. Thus we have a well defined morphism

$$T(f) : \quad \mathrm{Alb}(X)_{\mathbb{C}} \longrightarrow \mathrm{Jac}(X)_{\mathbb{C}}.$$

With CM-point P_χ in the last subsection, we can define the height pairing

$$\langle T(f)P_\chi, P_\chi\rangle \in \mathbb{C}.$$

It can be shown that the pairing does not depend on the choice of P. Thus we have defined a distribution

$$\mathrm{NT}_\chi \in \mathrm{Hom}(C_0^\infty(\mathrm{GL}_2(\widehat{\mathbb{Q}})/\mathbb{Q}^\times, \mathbb{C}), \quad \mathbb{C}).$$

Notice that the space $C_0^\infty(\mathrm{GL}_2(\widehat{\mathbb{Q}})/\mathbb{Q}^\times, \mathbb{C})$ is a representation of $\mathrm{GL}_2(\widehat{\mathbb{Q}})^2$ by action

$$(g_1, g_2)f(x) = f(g_1^{-1}xg_2), \qquad g_i \in \mathrm{GL}_2(\widehat{\mathbb{Q}}),\ x \in \mathrm{GL}_2(\widehat{\mathbb{Q}})/\mathbb{Q}^\times.$$

It is easy to see that NT_χ has character (χ^{-1}, χ) under the action of $\widehat{E}^\times \times \widehat{E}^\times$ (here χ is viewed as a character of $\widehat{E}^\times$ via class field theory):

$$\mathrm{NT}_\chi((t_1, t_2)f) = \chi^{-1}(t_1)\chi(t_2)\mathrm{NT}_\chi(f).$$

In the following we want to show that NT_χ factors through an automorphic quotient of weight 2.

Notice that $C_0^\infty(\mathrm{GL}_2(\widehat{\mathbb{Q}})/\mathbb{Q}^\times, \mathbb{C})$ acts on a representation σ of $\mathrm{GL}_2(\widehat{\mathbb{Q}})/\mathbb{Q}^\times$ by the formula

$$\sigma(f)v = \int_{\mathrm{GL}_2(\widehat{\mathbb{Q}})/\mathbb{Q}^\times} f(g)\sigma(g)v dg, \qquad f \in C_0^\infty(\mathrm{GL}_2(\widehat{\mathbb{Q}})/\mathbb{Q}^\times, \mathbb{C}), \quad v \in V_\sigma.$$

We fix such a Haar measure dg such that for any open compact subgroup U,

$$\mathrm{vol}(U) = (\mathrm{vol}(X_U))^{-1}$$

where $\mathrm{vol}(X_U)$ is calculated using measure $dxdy/(2\pi y^2)$. If σ is admissible, then we can show that the action of Hecke algebra defines a surjective morphism

$$C_0^\infty(\mathrm{GL}_2(\widehat{\mathbb{Q}})/\mathbb{Q}^\times, \mathbb{C}) \longrightarrow \sigma \otimes \widetilde{\sigma}$$

as representations of $\mathrm{GL}_2(\widehat{\mathbb{Q}})^2$. If we apply this to the representation S_2 on space of cusp forms of weight 2, then we obtain a surjective morphism

$$\rho_2 : \qquad C_0^\infty(\mathrm{GL}_2(\widehat{\mathbb{Q}})/\mathbb{Q}^\times, \mathbb{C}) \twoheadrightarrow \oplus_\pi \pi^\infty$$

where π runs through the set of cuspidal representations of weight 2.

An important property of NT_χ is that it factors the morphism ρ_2. In other words, there are

$$\mathrm{NT}_{\chi,\pi} \in \mathrm{Hom}(\pi^\infty \otimes \widetilde{\pi}^\infty, \mathbb{C})$$

for each cuspidal representation such that

$$\mathrm{NT}_\chi = (\oplus \mathrm{NT}_{\chi,\pi}) \circ \rho_2.$$

The functional $\mathrm{NT}_{\chi,\pi}$ also has character (χ^{-1}, χ) under the action by $(\widehat{E}^\times)^2$. Thus we have defined for each cusp representation π of weight 2 an functional:

$$\mathrm{NT}_{\chi,\pi} \in \mathrm{Hom}_{\widehat{E}^\times}(\pi^\infty \otimes \chi, \mathbb{C}) \otimes \mathrm{Hom}_{\widehat{E}^\times}(\widetilde{\pi}^\infty \otimes \chi^{-1}, \mathbb{C}).$$

3 Gross–Zagier and Waldspurger formulae

3.1 Local theory

In this subsection we want to study the local space

$$\mathrm{Hom}_{\widehat{E}^\times}(\pi^\infty \otimes \chi, \mathbb{C}).$$

The first task is look the bigger space

$$\mathrm{Hom}_{\widehat{\mathbb{Q}}^\times}(\pi^\infty \otimes \chi, \mathbb{C}) = \mathrm{Hom}_{\widehat{\mathbb{Q}}^\times}(V_\pi \otimes \omega_\pi^\infty \otimes \chi|_{\widehat{\mathbb{Q}}}, \mathbb{C}),$$

where V_π is the underlying space of π with trivial action by $\widehat{\mathbb{Q}}^\times$. It is clear this space is non-trivial only if the following condition holds:

$$\omega_\pi \cdot \chi|_{\widehat{\mathbb{Q}}^\times} = 1.$$

We will assume that this condition in the following discussion. Now we decompose the above linear space into local spaces:

$$\mathrm{Hom}_{\widehat{E}^\times}(\pi^\infty \otimes \chi, \mathbb{C}) = \otimes_p \mathrm{Hom}_{E_p^\times}(\pi_p \otimes \chi_p, \mathbb{C}),$$

and study the local space for each prime p:

$$\mathrm{Hom}_{E_p^\times}(\pi_p \otimes \chi_p, \mathbb{C}).$$

The first result is the following criterion:

Proposition 3.1.1 (Saito–Tunnel). *Assume that $\omega_p \chi|_{\mathbb{Q}_p^\times} = 1$. Then*

1. *if either π_p is principal or E_p is split then*

$$\dim \mathrm{Hom}_{E_p^\times}(\pi_p \otimes \chi_p, \mathbb{C}) = 1;$$

2. *if π_p is discrete and E_p is non-split then*

$$\dim \mathrm{Hom}_{E_p^\times}(\pi_p \otimes \chi_p, \mathbb{C}) + \dim \mathrm{Hom}_{E_p^\times}(\pi_p^{JL} \otimes \otimes \chi_p, \mathbb{C}) = 1$$

where π_p^{JL} is the Jacquet–Langlands correspondence of π_p on $D_p^\times$ for a division algebra D_p over $\mathbb{Q}_p$.

We want to explain a little bit about the Jacquet–Langlands correspondence. For each prime p, there is a quaternion division algebra D_p which is unique up to isomorphism. Moreover if E_p is a quadratic field extension of $\mathbb{Q}_p$ then there is a unique embedding $E_p \longrightarrow D_p$ of $\mathbb{Q}_p$-algebras which is unique up to conjugation by $D_p^\times$. In fact for any such embedding, we can write D_p in the following way

$$D_p = E_p + E_p j, \qquad jx = \bar{x} j, \forall x \in E_p, j^2 \in \mathbb{Q}^\times \setminus \mathrm{N}(E_p^\times).$$

When π_p is discrete, then there is a representation π_p^{JL} of $D_p^\times$, the Jacquet–Langlands correspondence of π which has he same central character as π_p and the *opposite character* as π_p:

$$\mathrm{tr}(g|\pi_p) = -\mathrm{tr}(g'|\pi_p^{JL})$$

for $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ regular and $g' \in D_p^\times$ with the same reduced trace and norm.

A precise criterion of Saito and Tunnel is to use root number of L-series $L(s, \pi, \chi)$, the Rankin–Selberg convolution of L-series $L(s, \pi)$ and $L(s, \chi)$. This L-series has a holomorphic continuation to whole complex plan and satisfies a functional equation

$$L(s, \pi, \chi) = \epsilon(s, \pi, \chi) L(1 - s, \widetilde{\pi}, \chi^{-1}) = \epsilon(s, \pi, \chi) L(1 - s, \pi, \chi).$$

The second equality follows from the condition $\omega_\pi \cdot \chi|_{\mathbb{A}^\times} = 1$. If we apply the functional twice then we obtain

$$\epsilon(s, \pi, \chi)\epsilon(1 - s, \pi, \chi) = 1.$$

This implies in particular that $\epsilon(1/2, \pi, \chi) = \pm 1$ which is called the root number of L-series $L(s, \pi, \chi)$. We can further decompose the ϵ-factor after fixing a additive character ψ of $\mathbb{Q}\backslash\mathbb{A}$:

$$\epsilon(s, \pi, \chi) = \prod_{p \le \infty} \epsilon(s, \pi_p, \chi_p, \psi_p).$$

The value at $1/2$ of these local ϵ factor which takes values in ± 1 and does not depend on the choice of ψ_p. We normalize the local root number as

$$\epsilon_p := \epsilon(1/2, \pi_p, \chi_p)\eta_p(-1)\chi_p(-1) = \pm 1$$

here $\eta = \otimes \eta_p$ is the quadratic character of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$ corresponding to the quadratic extension $E/\mathbb{Q}$.

Now we come to the second criterion of Saito and Tunnel

Proposition 3.1.2. *Assume* $\omega_p \cdot \chi|_{\mathbb{Q}_p} = 1$.

1. *if* $\epsilon_p = 1$, *then*
$$\dim \mathrm{Hom}_{E_p^\times}(\pi_p \otimes \chi_p, \mathbb{C}) = 1.$$
2. *if* $\epsilon_p = -1$, *then*
$$\dim \mathrm{Hom}_{E_p^\times}(\pi_p^{JL} \otimes \chi_p, \mathbb{C}) = 1.$$

3.2 Gross–Zagier formula for modular curves

With calculation in the last subsection we see that $\mathrm{NT}_{\pi,\chi} \neq 0$ only if

$$\mathrm{Hom}_{\widehat{E}^\times}(\pi^\infty \otimes \chi, \mathbb{C}) \otimes \mathrm{Hom}_{\widehat{E}^\times}(\widetilde{\pi}^\infty \otimes \chi^{-1}, \mathbb{C}) \neq 0.$$

The non-vanishing of the above space is equivalent to the following two conditions are hold:

1. $\omega_\pi \cdot \chi|_{\widehat{\mathbb{Q}}^\times} = 1$;
2. for all finite prime p, $\epsilon_p = 1$.

The last thing we want to discuss is about construction of concrete elements in

$$\mathrm{Hom}_{\widehat{E}^\times}(\pi^\infty \otimes \chi, \mathbb{C}) \otimes \mathrm{Hom}_{\widehat{E}^\times}(\widetilde{\pi}^\infty \otimes \chi^{-1}, \mathbb{C}).$$

In general it seems hard to construct any element in any one of the spaces appeared in the tensor product, but one obvious one element $\beta_{\pi,\chi}$ in the tensor product is

$$\beta_{\pi,\chi}(v \otimes \widetilde{v}) := \int_{\widehat{E}^\times/\widehat{\mathbb{Q}}^\times} \langle \pi^\infty(t)v, \widetilde{v}\rangle \chi(t)dt, \qquad v \otimes \widehat{v} \in \pi^\infty \otimes \widehat{\pi}^\infty$$

where dt is some Haar measure on $\widehat{E}^\times/\widehat{\mathbb{Q}}^\times$ and $\widehat{\mathbb{Q}}^\times$ is swiped out because of the condition $\omega \cdot \chi|_{\widehat{\mathbb{Q}}^\times} = 1$. Unfortunately, this integral is not convergent. To regularize this integral, we consider the local pairing:

$$\beta_{\pi_p,\chi_p}(v \otimes \widetilde{v}) = \int_{E_p^\times/\mathbb{Q}_p^\times} \langle \pi_p(t)v, \widetilde{v}\rangle \chi(t)dt, \qquad v \otimes \widetilde{v} \in \pi_p \otimes \widetilde{\pi}_p.$$

Then we have the following major result of Waldspurger:

Proposition 3.2.1 (Waldspurger)**.** *Define β_{π^{JL},χ_p} by the same formula as above when π_p is non-principal and E_p is non-split.*

1. *if $\epsilon_p = 1$ then $\beta_{\pi_p,\chi_p} \neq 0$;*
2. *if $\epsilon_p = -1$ then $\beta_{\pi_p^{JL},\beta_p} \neq 0$;*
3. *if everything is unramified then*

$$\beta_{\pi_p,\chi_p}(v \otimes \widetilde{v}) = \frac{\zeta_p(2)L(1/2, \pi_p, \chi_p)}{L(1, \eta_p)L(1, \pi_p, ad)}.$$

We need some explanation for item 3 in the proposition. First of all "*Everything is unramified here*" means the following

- π_p is unramified and χ_p on $E_p^\times$ is unramfied;
- $v \otimes \widetilde{v} \in \pi_p^{\mathrm{GL}_2(\mathbb{Z}_p)} \otimes \widetilde{\pi}_p^{\mathrm{GL}_2(\mathbb{Z}_p)}$, and $\langle v, \widetilde{v}\rangle = 1$;
- the Haar measure on $E_p^\times/\mathbb{Q}_p^\times$ is chosen such that $\mathrm{vol}(\mathscr{O}_{E_p}^\times/\mathbb{Z}_p^\times) = 1$.

Secondly, we need to explain the L-series in the right hand side of formula:

$$\zeta_p(s) = \frac{1}{1 - p^s}, \qquad L(s, \eta_p) = \frac{1}{1 - \eta_p(p)p^s},$$

and $L(s, \pi_p, \chi_p)$ and $L(1, \pi_p, \mathrm{ad})$ can be written in terms of L-series of π_p and χ_p. More precisely, let

$$L(s, \pi_p) = \frac{1}{(1 - \alpha_p p^s)(1 - \beta_p p^s)}, \qquad L(s, \chi_p) := \prod_{\wp|p} L(s, \chi_\wp) = \frac{1}{(1 - \alpha_p' p^s)(1 - \beta' p^s)}$$

then

$$L(s, \pi_p, \chi_p) = \frac{1}{(1 - \alpha_p\alpha_p' p^s)(1 - \alpha_p\beta_p' p^s)(1 - \beta_p\alpha_p' p^s)(1 - \beta_p\beta_p' p^s)},$$

$$L(s,\pi_p,ad)=\frac{1}{1-\alpha_p\beta_p^{-1}p^s)(1-p^s)(1-\beta_p\alpha_p^{-1}p^s)}.$$

Since the numbers in the right side in item 3 in the above proposition is well defined and nonzero for all p, we may normalize the local pairing by introduce

$$\alpha_{\pi_p,\chi_p}:=\frac{L(1,\eta_p)L(1,\pi_p,ad)}{\zeta_p(2)L(1/2,\pi_p,\chi_p)}\beta_{\pi_p,\chi_p}.$$

Similarly, we can define $\alpha_{\pi_p^{JL},\chi_p}$. Now the global pairing $\beta_{\pi,\chi}$ can be regularized by the following formula:

$$\beta_{\pi,\chi}=\frac{\zeta(2)}{L(1,\eta)L(1,\pi,ad)}\prod_p\alpha_{\pi_p,\chi_p}.$$

Of course the factor here is defined by analytic continuation of various L-series. It is finite and positive.

Theorem 3.2.2 (Gross–Zagier, Yuan–Zhang–Zhang)**.** *Assume that $\omega_\pi\cdot\chi|_{\widehat{\mathbb{Q}}^\times}=1$ and that $\epsilon_p=1$ for all prime p. Then $L(1/2,\pi,\chi)=0$, and*

$$\mathrm{NT}_{\pi,\chi}=L'(\frac{1}{2},\pi,\chi)L(1,\eta)\beta_{\pi,\chi}$$

in

$$\mathrm{Hom}_{\widehat{E}^\times}(\pi^\infty\otimes\chi,\mathbb{C})\otimes\mathrm{Hom}_{\widehat{E}^\times}(\widetilde{\pi}^\infty\otimes\chi^{-1},\mathbb{C}).$$

The vanishing of $L(1/2,\pi,\chi)$ follows from the fact that the sign of the functional equation is $\epsilon(1/2,\pi,\chi)$. In fact the condition implies that that $\Sigma=\{\infty\}$ and that

$$\epsilon(1/2,\pi,\chi)=(-1)^{\#\Sigma}=-1.$$

3.3 Gross–Zagier formula for Shimura curves over $\mathbb{Q}$

Let π be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ of weight 2 such that its central character ω_π has trivial component at infinity. Let $E/\mathbb{Q}$ be an imaginary quadratic field and χ be a character of $E^\times\backslash\mathbb{A}_E^\times$ such that $\omega_\pi\cdot\chi|_{\mathbb{A}^\times}=1$. Then we have a functional equation

$$L(s,\pi,\chi)=\epsilon(s,\pi,\chi)L(1-s,\pi,\chi).$$

At $s=1/2$, $\epsilon(1/2,\pi,\chi)=\pm1$ can be written as

$$\epsilon(1/2,\pi,\chi)=(-1)^{\#\Sigma}.$$

Here Σ is a finite set including archiemdean place. Assume that the sign is -1, then Σ is odd and $L(1/2,\pi,\chi)=0$. The Gross–Zagier formula in the last section gives an expression for $L'(1/2,\pi,\chi)$ in terms of height pairing of CM-points when $\Sigma=\{\infty\}$. What can we say about general case of odd Σ? The answer is to use Shimura curve.

Since Σ is odd, there is an (indefinite) quaternion algebra B over $\mathbb{Q}$ ramified exactly places at p:

$$B_p\simeq D_p\Leftrightarrow p\in\Sigma\setminus\{\infty\}.$$

In particular, at the arhimedean place $B \otimes \mathbb{R} \simeq M_2(\mathbb{R})$. Fix such an isomorphism, then $B_+^\times$ acts on $\mathscr{H}$. A subgroup Γ of $B_+^\times$ is called a congruence subgroup if there is a maximal order $\mathscr{O}_B$ and a number N such that

$$(1 + N\mathscr{O}_B) \cap \mathscr{O}_{B,+}^\times \subset \Gamma \subset \mathscr{O}_{B,+}^\times.$$

For such a Γ, we can form a Riemann surface

$$X_\Gamma = \Gamma \backslash \mathscr{H}.$$

Such a curve is compact if $\Sigma \neq \{\infty\}$, or $B \neq M_2(\mathbb{Q})$ which assume in the following discussion. Again we consider the projective system X of such curves which admits an right action by $B^\times/\mathbb{Q}^\times$. Also every such a curve has a canonical model defined over $\bar{\mathbb{Q}}$ and all Galois conjugates of these curves are also Shimura curve in the same type. To understand the Galois conjugates, it is better to use adeles which we will do as follows.

For each open and compact subgroup U of $\widehat{B}^\times$, define the curve

$$X_U = B^\times \backslash \mathscr{H}^\pm \times \widehat{B}^\times / U.$$

Then X_U is canonically defined over $\mathbb{Q}$ and parameterizes the equivalent classes of the triple (A, i, κ) where

1. A is an abelian surface;
2. $i : B \longrightarrow \mathrm{End}(A) \otimes \mathbb{Q}$;
3. $\kappa : \widehat{B} \longrightarrow \widehat{V}(A)$ a $\widehat{B}$-linear isomorphism modulo U which acts on $\widehat{B}$.

The connected component of X_U is identified with

$$B^\times \backslash \{\pm 1\} \times \widehat{B}^\times / U \simeq B_+^\times \backslash \{\pm 1\} \times \widehat{B}^\times / U \simeq \mathbb{Q}_+^\times \backslash \widehat{\mathbb{Q}}^\times / \nu(U) \simeq \widehat{\mathbb{Z}}^\times / \nu(U)$$

where the second isomorphism is given by reduced norm. We may also decompose X_U into a finite disjoint union of connected curves X_Γ.

For imaginary quadratic field E and any embedding $\widehat{E} \subset \widehat{B}$, we may consider subscheme $X^{E^\times}$ of points fixed by $E^\times$. Such a scheme has an action by the normalizer of $E^\times$ in $\widehat{B}^\times$ which is $\widehat{E}^\times \coprod \widehat{E}^\times j$ where $j \in \widehat{B}^\times$ is an element in $\widehat{B}^\times$ such that $jx = \bar{x}j$ and $j^2 \in \widehat{\mathbb{Q}}^\times$. Since $E^\times$ acts trivially on this space, the action factor through the quotient group

$$H = E^\times \backslash (\widehat{E}^\times \coprod \widehat{E}^\times j).$$

The theory of complex multiplication says the following:

1. the set $X^{E^\times}(\bar{\mathbb{Q}})$ is a principal homogenous space of H;
2. every point in $X^{E^\times}(\bar{\mathbb{Q}})$ is actually defined over E^{ab};
3. that action of $\mathrm{Gal}(E^{\mathrm{ab}}/\mathbb{Q})$ on $X^{E^\times}(\bar{\mathbb{Q}})$ is given by the class field theory isomorphism:

$$\mathrm{Gal}(E^{\mathrm{ab}}/\mathbb{Q}) \simeq H.$$

Now pick up any point $P \in P^{E^\times}(\bar{\mathbb{Q}})$ and define

$$P_\chi = \int_{\mathrm{Gal}(E^{\mathrm{ab}}/E)} \chi^{-1}(\sigma)(P^\sigma - \xi_{P^\sigma})d\sigma \in \mathrm{Alb}(X)(E^{\mathrm{ab}}) \otimes \mathbb{C}.$$

Here we identify χ as a character of Galois group via class field theory, and that ξ_{P^σ} is the Hodge bundle on the connected component of X containing P^σ.

Now for any function $f \in C_0^\infty(\widehat{B}^\times/\mathbb{Q}^\times)$ we can define a Heck correspondence $T(f)$ as in the modular curve case. Thus we obtain an element

$$T(f)P_\chi \in \mathrm{Jac}(X)(E^{\mathrm{ab}}) \otimes \mathbb{C}.$$

Now we can form the Neron–Tate pairing

$$\mathrm{NT}_\chi(f) := \langle T(f)P_\chi, P_\chi\rangle \in \mathbb{C}.$$

Again this distribution has character (χ^{-1}, χ) under the action of $\widehat{E}^\times \times \widehat{E}^\times$ and factors through the Hecke action on the space of cusp form of weight 2:

$$C_0^\infty(\widehat{B}^\times/\mathbb{Q}^\times) \twoheadrightarrow \oplus\sigma^\infty \otimes \widetilde{\sigma}.$$

Thus we have a well-defined pairing for each irreducible $B_{\mathbb{A}}^\times$ cuspidal representation σ of weight 2:

$$\mathrm{NT}_{\sigma,\chi} \in \mathrm{Hom}_{\widehat{E}^\times}(\sigma \otimes \chi, \mathbb{C}) \otimes \mathrm{Hom}_{\widehat{E}^\times}\widetilde{\sigma} \otimes \chi^{-1}, \mathbb{C}).$$

By our construction of B, we know that following:

1. one of σ is π^{JL};
2. for π^{JL}, the space of in the right hand side in the above formula is non-zero. Moreover, it has a canonical element $\beta_{\pi^{JL},\chi}$.

The Gross–Zagier formula proved in this case is as follows:

Theorem 3.3.1 (Gross–Zagier, Yuan–Zhang–Zhang)**.** *With the above notation*

$$\mathrm{NT}_{\pi^{JL},\chi} = L'(1/2, \pi, \chi)L(1, \eta)\beta_{\pi^{JL},\chi}.$$

3.4 Waldspurger formula

In the Gross–Zagier formula, we only treat the case where the sign of the functional equation is -1. What about $+1$? This case is solved a formula of Waldspurger. In fact, since there is no algebraic geometry involved, Waldspurger's formula holds in a great generality which we will describe as follows.

Let F be a number field and π an irreducible cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Let E/F be a quadratic extension and χ a quasi-character of $E^\times\backslash\mathbb{A}_E^\times$. Then we have a Rankin–Selberg L-series $L(s, \pi, \chi)$ as a holomorphic function of $s \in \mathbb{C}$ which satisfies a functional equation

$$L(s, \pi, \chi) = \epsilon(s, \pi, \chi)L(1 - s, \widetilde{\pi}, \chi^{-1}).$$

Assume that the central character ω of π and χ satisfy

$$\omega \cdot \chi|_{\mathbb{A}_F^\times} = 1$$

then the above functional equation is symmetric:

$$L(s, \pi, \chi) = \epsilon(s, \pi, \chi)L(1 - s, \pi, \chi).$$

The root number $\epsilon(1/2, \pi, \chi)$ takes values ± 1 and is again a product of local root numbers:

$$\epsilon(1/2, \pi, \chi) = (-1)^{\#\Sigma}$$

$$\Sigma = \{v : \epsilon(1/2, \pi_v, \chi_v, \psi_v)\eta_v\chi_v(-1) = -1.\}$$

Here $\psi = \prod \psi_v$ is a character of $F\backslash\mathbb{A}_F$ is an additive character used to decompose ϵ-factor into a product of local factors but whose values at 1/2 does not depend on the choice of ψ_v.

Assume that the sign of the functional equation is +1 then Σ is even and contains no complex places of F or finite places split in E. Let B be a quaternion algebra over F with ramification set Σ. Then π will have a Jacquet–Langlands correspondence π^{JL} as an irreducible sub-representation of $B_\mathbb{A}^\times$ in $L^2(B^\times\backslash B_\mathbb{A}^\times, \omega)$. Fix any embedding of E into B whose existence is guaranteed by chosen of B. Then we can define a distribution on $\pi^{JL} \otimes \widetilde{\pi}^{JL}$:

$$\mathrm{Walds}_{\pi^{JL},\chi}(f\otimes\widetilde{f}) = \int_{E^\times\mathbb{A}_F^\times\backslash E_\mathbb{A}^\times} f(t)\chi(t)dt\cdot\int_{E^\times\mathbb{A}_F^\times\backslash E_\mathbb{A}^\times} \widetilde{f}(t)\chi^{-1}(t)dt, \qquad f\otimes\widetilde{f} \in \pi^{JL}\otimes\widetilde{\pi}^{JL}.$$

It is clear that

$$\mathrm{Walds}_{\pi^{JL},\chi} \in \mathrm{Hom}_{E_\mathbb{A}^\times}(\pi^{JL} \otimes \chi, \mathbb{C}) \otimes \mathrm{Hom}_{E_\mathbb{A}^\times}(\widetilde{\pi}^{JL} \otimes \chi^{-1}, \mathbb{C}).$$

Under our construction of B, the last space in the above formula has a non-zero element $\beta_{\pi^{JL},\chi}$. The main result of Waldspurger is

Theorem 3.4.1 (Waldspurger)**.**

$$\mathrm{Walds}_{\pi^{JL},\chi} = \frac{1}{2}L(1/2, \pi, \chi)L(1, \eta)\beta_{\pi^{JL},\chi}.$$

4 Gross–Zagier formula over totally field and applications

In this section, we fix a totally number field F and write $\mathbb{A}$ for the adeles for F.

4.1 Shimura curves over totally real field

Let Σ be a finite set of places of F. If Σ is even, there is a unique quaternion algebra B over F with ramification set Σ. If Σ is odd, then there is no such quaternion algebra over F, but there is a quaternion algebra $\mathbb{B}$ over $\mathbb{A}$ with ramification set Σ, i.e., as a module over $\mathbb{A}$, $\mathbb{B}$ is free of rank 4 and for each place v, $\mathbb{B}_v := \mathbb{B} \otimes_{\mathbb{A}} F_v$ is isomorphic to $M_2(F_v)$ if v is not in Σ and to the division quaternion algebra D_v of $v \in \Sigma$. Such a quaternion algebra over $\mathbb{B}$ is called *incoherent* since it is not comes from a quaternion algebra over F.

Assume that Σ is odd and including all archimedean places of F. Then for each open and compact subgroup U of $\mathbb{B}_f^\times$, we have a Shimura curve X_U defined over F such that for each embedding $\sigma : F \longrightarrow \mathbb{R}$, the associated analytic curve can be written as in a usual way:

$$X_{U,\sigma}^{\mathrm{an}} =_\sigma B^\times \backslash \mathscr{H}^\pm \times \mathbb{B}_f^\times / U.$$

Here ${}_\sigma B$ is a quaternion algebra over F with ramification set $\Sigma \setminus \{\sigma\}$ with a fixed isomorphism $B \otimes_\sigma \mathbb{R} = M_2(\mathbb{R})$ and $\widehat{B} \simeq \mathbb{B}_f$. When U various, the curves X_U form a projective system with an action of $\mathbb{B}^\times$. We write X for the projective limit, and let $\mathbb{B}^\times$ act on X so that $\mathbb{B}_\infty^\times$ trivially on X. Then it is easy to see that the subgroup of $\mathbb{B}^\times$ acting trivially on X is $D := \mathbb{B}_\infty^\times \cdot \overline{F^\times}$ where $\overline{F^\times}$ is the topological closure of $F^\times$ in $\mathbb{A}^\times / F_\infty^\times$.

We may define the Jacobian $\mathrm{Jac}(X)$ and the $\mathrm{Alb}(X)$ as in the same way as in modular curve or Shimura curve over $\mathbb{Q}$. The Jacobian has a decomposition up to isogeny as follows

$$\mathrm{Jac}(X) \otimes \mathbb{Q} = \oplus_\Pi \Pi \otimes_{L_\Pi} (A_\Pi \otimes \mathbb{Q})$$

where Π run through a set of the irreducible $\mathbb{Q}$ representation of $\mathbb{B}^\times / D$ with endomorphic field L_Π, and A_Π is an abelian variety of F with endomorphic by an order of L_Π. More precisely, Π corresponding bijectively to the set of Galois conjugate classes of cuspidal representations $\{\pi\}$ of $\mathrm{GL}_2(\mathbb{A})$ of parallel weight 2 which is discrete at places in Σ. Here we say that two such representations π_1 and π_2 are Galois conjugate to each other if there is an automorphism τ of $\mathbb{C}$ such that

$$\pi_1^\infty = \pi_2^\infty \otimes_\tau \mathbb{C}.$$

Via Jacquet–Langlands correspondence, such a representation gives a representation of $\mathbb{B}^\times / D$ with same Galois conjugate relation. The Abelian variety A_Π is characterized by the identity:

$$L(s, A_\Pi) = \prod_{\sigma \in \Pi} L(s - 1/2, \pi).$$

4.2 Gross–Zagier formula over totally fields

Let E be a totally imaginary quadratic extension of F and let χ be a finite character of $E^\times \backslash \mathbb{A}_E^\times$ such that

$$\omega_\pi \cdot \chi|_{\mathbb{A}_F^\times} = 1.$$

Then we have a Rankin–Selberg L-series $L(s, \pi, \chi)$ with functional equation

$$L(s, \pi, \chi) = \epsilon(s, \pi, \chi) L(1 - s, \pi, \chi).$$

The root number $\epsilon(1/2, \pi, \chi) = \pm 1$ and is the product of local root numbers $\epsilon(1/2, \pi_v, \chi_v, \psi_v) = \pm 1$ via an additive character ψ of $F\backslash \mathbb{A}_F$ but does not depend on the choice of ψ. Thus we have

$$\epsilon(1/2, \pi, \chi) = (-1)^{\#\Sigma}$$

where

$$\Sigma = \{v : \quad \epsilon(1/2, \pi_v, \chi_v, \psi_v)\eta_v\chi_v(-1) = -1\}$$

where η is a quadratic character of $F^\times\backslash\mathbb{A}_F^\times$ corresponding to extension E/F.

Assume that Σ is odd and let X be the Shimura variety defined an incoherenet quaternion algebra $\mathbb{B}$ with ramification set Σ defined as in the last subsection. Fix any embedding $\mathbb{A}$-embedding $\mathbb{A}_E \longrightarrow \mathbb{B}$ with is unique up to conjugation by $\mathbb{B}^\times$. Let $X^{E^\times}$ be the subcheme over F fixed by $E^\times$. Then $X^{E^\times}$ has an action by the normalizer of $E^\times$ in $\mathbb{B}^\times$ which is generated by $\mathbb{A}_E^\times$ and an element $j \in \mathbb{B}^\times$ such that $jx = \bar{x}j$ for all $x \in \mathbb{A}_E$ and that $j^2 \in \mathbb{A}^\times$. Let H be the quotient of this group modulo $\overline{E^\times} \cdot H_\infty^\times$ which acts trivially on $X^{E^\times}$ where $\overline{E^\times}$ is the closure of $E^\times$ in $E_\infty^\times\backslash\mathbb{A}_E^\times$. The Shimura theory says the following

1. every point $X^{E^\times}(\bar{F})$ is defined over E^{ab};
2. $X^{E^\times}(E^{\mathrm{ab}})$ is a principal homogenous space of H;
3. the action of $\mathrm{Gal}(E^{\mathrm{ab}}/F)$ on $X^{E^\times}(E^{\mathrm{ab}})$ is given by an isomorphism

$$\mathrm{Gal}(E^{\mathrm{ab}}/F) \simeq H$$

defined in the class field theory.

Let $P \in X^{E^\times}(E^{\mathrm{ab}})$ be any point and let

$$P_\chi = \int_{\mathrm{Gal}(E^{\mathrm{ab}}/E)} \chi(\sigma)^{-1}(P^\sigma - \xi_{P^\sigma})d\sigma \in \mathrm{Alb}(X)(E^{\mathrm{ab}}) \otimes \mathbb{C}$$

with respect to a Haar measure of volume $2L(1, \eta)$, where $\xi_{P^\times}$ is the Hodge class in the connected component of X containing P^σ.

For any function $f \in C_0^\infty(\mathbb{B}^\times/D, \mathbb{C})$ one can define a class $Z(f) \in \mathrm{Ch}^1(X \times X)$, the direct limit of $\mathrm{Ch}^1(X_U \times X_U)$. Then one can define

$$T(f)P_\chi := Z(f)_* P_\chi \in \mathrm{Jac}(X)(E^{\mathrm{ab}}) \otimes \mathbb{C}.$$

The Neron–Tate pairing gives a distribution

$$\mathrm{NT}_\chi \in \mathrm{Hom}(C_0^\infty(\mathbb{B}^\times/D, \mathbb{C}), \mathbb{C}), \qquad f \mapsto \langle T(f)P_\chi, P_\chi\rangle$$

where the pairing is hermitain in $\mathbb{C}$.

Consider $C_0^\infty(\mathbb{B}^\times/D, \mathbb{C})$ as a representation of $\mathbb{B}^\times \times \mathbb{B}^\times$ by left and right translation. One may again prove that NT_χ has translation property under $\mathbb{A}_E^\times \times \mathbb{A}_E^\times$ by

$$\mathrm{NT}_\chi((t_1, t_2)f) = \chi^{-1}(t_1)\chi(t_2)\mathrm{NT}_\chi(f),$$

and that the distrbution factors through the surjective morphism

$$C_0^\infty(\mathbb{B}^\times/D, \mathbb{C}) \longrightarrow \oplus_\sigma \sigma \otimes \widetilde{\sigma}$$

where σ runs though the Jacquet–Langlands correspondences of irreducible cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ to $\mathbb{B}^\times$. Thus we have define a pairing for each such a σ:

$$\mathrm{NT}_{\sigma,\chi} \in \mathrm{Hom}_{\mathbb{A}_E^\times}(\sigma \otimes \chi, \mathbb{C}) \otimes \mathrm{Hom}_{\mathbb{A}_E^\times}(\widetilde{\sigma} \otimes \chi^{-1}, \mathbb{C}).$$

We take $\sigma = \pi^{JL}$. Then the right hand side of the above expression is one dimensional and is generated by an element $\beta_{\pi^{\mathrm{JL}},\chi}$ regularize the integral

$$\int_{E^\times \mathbb{A}^\times \backslash \mathbb{A}_E^\times} \langle \pi^{JL}(t)v, \widetilde{v}\rangle dt$$

via a regularization process.

Theorem 4.2.1 (Gross–Zagier, Yuan–Zhang–Zhang)**.**

$$\mathrm{NT}_{\pi^{JL},\chi} = L'(1/2, \pi, \chi)L(1, \eta)\beta_{\pi^{\mathrm{JL}},\chi}.$$

4.3 Application to the Birch and Swinnerton-Dyer conjecture

Let π be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ of parallel weight 2. Let $\Pi = \{\pi^\sigma\}$ be the Galois conjugate class of π. Then it is conjectured that there is an abelian variety A_Π with endomorphism by L_Π the definition field of Π, and with L- series

$$L(s, A_\Pi) = \prod_{\pi^\sigma \in \Pi} L(s - 1/2, \pi^\sigma).$$

If π is of CM-type, i.e., the lifting from a Hecke character of an imaginary quadratic extension of F, then A_π can be constructed as an CM-abelian variety. If $[F : \mathbb{Q}]$ is odd or one of local component π_v is not principal, then we can construct A_Π from Jacobian of Shimura variety as above. Besides these two cases, there is no known construction of A_π. But a compatible system of ℓ-adic representation has been constructed by Taylor.

Let E/F be an imaginary quadratic extension and let χ be a finite character of $E^\times \backslash \mathbb{A}_E^\times$ such that

$$\omega_\pi \cdot \chi|_{\mathbb{A}^\times} = 1.$$

Then $L(s, \pi\chi)$ has root number $\epsilon(1/2, \pi, \chi) = (-1)^{\#\Sigma}$. In case of Σ is odd, the abelian variety A_Π can be constructed from Jacobian of the Shimura curve defined by an incoherent quaternion algebra with ramification set Σ. If Σ is even such a construction does not work.

Using Kolyvagin's Euler system technique, we can prove the following consequence of the Birch and Swinner-Dyer conjecture:

Theorem 4.3.1 (Tian–Zhang)**.** *Assume that* $\mathrm{ord}_{s=1}L(s, \pi, \chi) = 1$*. Then*

1. $\mathrm{ord}_{s=1}L(s, \pi^\sigma, \chi^\sigma) = 1$ *for all automorphism* σ *of* $\mathbb{C}$*;*
2. $\dim(A_\Pi(E^{\mathrm{ab}}) \otimes \mathbb{C})^\chi = \deg L_\Pi(\chi)$ *where* $(A_\Pi(E^{\mathrm{ab}}) \otimes \mathbb{C})^\chi$ *is the* χ*-eigen subspace of* $A_\Pi(E^{\mathrm{ab}}) \otimes \mathbb{C}$ *under action by* $\mathrm{Gal}(E^{\mathrm{ab}}/E)$*; and* $L_\Pi(\chi)$ *is he extension of* L_Π *by adding all values of* χ*.*

3. *The* χ*-component of the Tate–Shafarevich group* $\text{Ш}(A_\Pi, \chi)$ *is finite.*

Combining with a results of Bump–Friedberg–Hoffstein, we have the following

Theorem 4.3.2 (Tian–Zhang). *Assume that* $\text{ord}_{s=1}L(s, \pi) \leq 1$ *and that* $[F : \mathbb{Q}]$ *is odd or one finite* π_v *is not principal. Then*

1. $\text{ord}_{s=1}L(s, \pi^\sigma) = \text{ord}_{s=1}L(s, \pi)$ *for all automorphism* σ *of* $\mathbb{C}$*;*
2. $\dim(A_\Pi(F)) = \deg L_\Pi \cdot \text{ord}_{s=1}L(s, \pi^\sigma)$
3. *The Tate–Shafarevich group* $\text{Ш}(A_\Pi)$ *is finite.*

Using congruence of modular forms we can also handle some case of rank 0 case not covered by the above Theorem:

Theorem 4.3.3 (Tian–Zhang). *Assume that* $\text{ord}_{s=1}L(s, \pi, \chi) = 0$ *and that* A *is not of CM-type. Then*

1. $\text{ord}_{s=1}L(s, \pi^\sigma, \chi^\sigma) = 0$ *for all automorphism* σ *of* $\mathbb{C}$*;*
2. $\dim(A_\Pi(E^{\text{ab}}) \otimes \mathbb{C})^\chi = 0$.
3. *If* χ *is trivial and* A *is geometrically connected, then* $\text{Ш}(A_\Pi)$ *is finite.*

5 Gross–Schoen cycles and triple product L-series

5.1 Gross–Schoen cycles

Let X be a projective, smooth, and geometrically connected curve defined over a field k. Let $e = (e_1, e_2, e_3) \in X(k)$ be a triple of rational points on X. The Gross–Schoen cycles Δ_e with base e is defined as the following 1-cycle on the triple product X^3 of X:

$$\begin{aligned}\Delta_e =& \{(x, x, x) | x \in X\} - \{(e_1, x, x)\} - \{x, e_2, x\} - \{(x, x, e_3)\} \\ &+ \{(x, e_2, e_3)\} + \{(x, e_2, e_3)\} + \{(e_1, e_2, x)\}.\end{aligned}$$

Lemma 5.1.1. *The class* Δ_e *is homologically trivial.*

Proof. We need to prove that the class of Δ_e vanishes in any Weil cohomology theory $H^*(X)$. For example, in the ℓ-adic cohomology for ℓ different than the characteristic of k. For this it suffices to show that it has zero pairing with any class $\alpha \in H^2(X^3)$. Using Künneth formula

$$H^2(X^3) = \oplus_{i+j+k=2} H^i(X) \otimes H^j(X) \otimes H^k(X)$$

one see any such a class is a sum of pull-back of forms $\alpha_{i,j}$ via projections $\pi_{i,j} : X^3 \longrightarrow X^2$:

$$\alpha = \sum_{i,j} \pi_{i,j}^* \alpha_{i,j}.$$

The the pairing has the decomposition

$$\langle \Delta_e, \alpha \rangle = \sum_{i,j} \langle \Delta_e, \pi_{ij}^* \alpha_{i,j} \rangle = \sum_{i,j} \langle \pi_{ij*} \Delta_e, \alpha_{ij} \rangle.$$

Now the vanishing of the pairing follows from the fact that $\pi_{ij*}\Delta_e = 0$ for any projection π_{ij}. □

Now the question is: *if* Δ_e *already zero in* $\mathrm{Ch}^2(X^3) \otimes \mathbb{Q}$*?* In fact when $X = \mathbb{P}^1$ or when X is a hyperelliptic curve such that e_i are all Weierstrass points, then Gross and Schoen shows that $\Delta_e = 0$ in $\mathrm{Ch}^2(X^3)_{\mathbb{Q}}$. In general it is not to see if this class is zero in the Chow group. In the following we want to introduce the heights of these points when k is a number field.

Replacing k by an extension, we may assume that X/k has a semistable model $\mathscr{X}$ over the ring $\mathscr{O}_k$ of integers; that is the minimal regular model such that only singular points in the fiber of the morphism $\mathscr{X} \longrightarrow \mathrm{Spec}\mathscr{O}_k$ are of ordinary singular points, namely with local equation $xy = \pi$ is formal neighborhoods of singular points of the fibers.

Now we consider the fiber triple product

$$\mathscr{X}^3 := \mathscr{X} \times_{\mathrm{Spec}\mathscr{O}_k} \mathscr{X} \times_{\mathrm{Spec}\mathscr{O}_k} \mathscr{X}$$

which is not regular if $\mathscr{X}$ is not smooth over $\mathrm{Spec}\mathscr{O}_k$. Then we blow-up $\mathscr{X}^3$ successively along the vertical hyper surfaces in the bad fiber of $\mathscr{Y} \longrightarrow \mathrm{Spec}\mathscr{O}_k$ with any fixed ordering. Then we obtain a regular four fold $\mathscr{Y}$. On this regular scheme, Gross–Schoen construct a nice extension:

Proposition 5.1.1 (Gross–Schoen)**.** *There is a cycle* $\widetilde{\Delta}_e$ *on* $\mathscr{Y}$ *extending* Δ_e *such that for any* 2*-cycle* Z *of* $\mathscr{Y}$ *included in a fiber*

$$\widetilde{\Delta}_e \cdot Z = 0.$$

To define the hight pairing, we still need a green current on $\mathscr{Y}(\mathbb{C}) = \coprod_{\sigma: k \longrightarrow \mathbb{C}} X_\sigma(\mathbb{C})^3$. Since Δ_e is cohomologically trivial, there is a green's current G of degree $(1,1)$ such that

$$\frac{\partial\bar{\partial}}{\pi i} G = \delta_{\Delta_e(\mathbb{C})}.$$

Now we can form a cycle $\widehat{\Delta}_e = (\widehat{\Delta}_e, G)$ in the arithmetical Chow group $\widehat{\mathrm{Ch}}^2(\mathscr{Y})$ and we can define the height:

$$h(\Delta_e) = \widehat{\Delta}_e \cdot \widehat{\Delta}_e.$$

This height is independent of choice of $\widehat{\Delta}_e$ and it is zero if Δ_e is vanishing in $\widehat{\mathrm{Ch}}^2(\mathscr{Y}) \otimes \mathbb{Q}$. Conversely, the arithmetical index conjecture of Beilin–Bloch and Gillet–Soule says that $h(\Delta_e) \geq 0$ and equal to 0 exactly when Δ_e vanishes in the Chow group.

Assume that $e_1 = e_2 = e_3$ then the height $h(\Delta_e)$ can be computed in terms of a canonical cycle Δ_ξ which we defined as follows when $g(X) > 1$. Let us denote

$$\xi = \omega_X/(2g-2) \in \mathrm{Pic}(X) \otimes \mathbb{Q}$$

as the canonical class with rational coefficients of degree 1. After an extension, we have a representative

$$\xi = \sum a_i p_i, \qquad p_i \in X(k)$$

and then define

$$\Delta_\xi = \sum_{i,j,k} a_i a_j a_k \Delta_{p_i, p_j, p_k}.$$

The class $\Delta_\xi \in \mathrm{Ch}^2(X^3)_{\mathbb{Q}}$ does not dependent of choice of presentation.

Assume that $e_1 = e_2 = e_3$. Then we can show that

$$h(\Delta_e) = h(\Delta_\xi) + 6(g-1)h(e_1 - \xi)$$

where $h(e_1 - \xi)$ is the Neron–Tate height of $e_1 - \xi$. Thus $h(\Delta_e)$ reaches its minimal exactly at Δ_ξ.

For Δ_ξ, we have the following identity:

$$h(\Delta_\xi) = \frac{2g+1}{2g-2}\omega_a^2 - \sum c(X_v)$$

where ω_a^2 is the self-intersection of the admissible relative dualising sheaf and $c(X_v)$ some local invariants. This is a very interesting identity because ω_a^2 is a very important invariant in diophantine geometry and $h(\Delta_\xi)$ is an important invariant in the cohomology theory. For example, a nice upper-bound of ω_a^2 will imply the ABC conjecture and a lower bound will imply an effective Bogomolov conjecture. On Δ_ξ part, its heights related to L-series via Beilinson–Bloch conjecture and its class in a Selmer group will relate the extension defined in the uniportant fundamental group

$$0 \longrightarrow \frac{[\pi_1, \pi_1]}{[\pi_1, [\pi_1, \pi_1]]} \longrightarrow \frac{\pi_1}{[\pi_1, [\pi_1, \pi_1]]} \longrightarrow \frac{\pi_1}{[\pi_1, \pi_1]} \longrightarrow 0.$$

We conclude this subsection by a remark that for any correspondence $Z \in \mathscr{H}^3(X^3 \otimes X^3)$, we can define pairing

$$\langle Z_* \Delta_\xi, \Delta_\xi \rangle.$$

5.2 Triple product L-series

Let F be a number field and let $E = F \oplus F \oplus F$ be the triple cubic semisimple extension of F. Though everything has analogue for non-trivial cubic extension, we take split E for simplicity. Let π be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_E)$ then we can define a triple L-series $L(s, \pi, r_8)$ of degree 8 by work of Garret, and Piatetski-Shapiro–Rallis which is holomoprhic with a functional equation

$$L(s, \pi, r_8) = \epsilon(s, \pi, r_8) L(1-s, \widetilde{\pi}, r_8).$$

Here r_8 denote the 8-dimension representation of the dual group $\mathrm{GL}_2(\mathbb{C})^8$ via tensor product: $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathbb{C}^8$.

At a finite place v where all three components π_1, π_2, and π_3 of π are unramified, this L-series can be defined easily as follows: write $L(s, \pi_{iv}) = \frac{1}{\det(1 - A_i \mathrm{N}v^{-s})}$ with $A_i \in \mathrm{GL}_2(\mathbb{C})$, then

$$L(s, \pi_v) = \frac{1}{\det(1 - A_1 \otimes A_2 \otimes A_3 \mathrm{N}v^{-s})}$$

where $A_1 \otimes A_2 \otimes A_3$ acts on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathbb{C}^8$.

Assume that the central character ω_π on $\mathbb{A}_E^\times = (\mathbb{A}_F^\times)^3$ is trivial on the diagonal, that is

$$\omega_{\pi_1} \omega_{\pi_2} \omega_{\pi_3} = 1.$$

Then the functional equation is symmetric

$$L(s, \pi, r_8) = \epsilon(s, \pi, r_8) L(1 - s, \pi, r_8).$$

It follows that $\epsilon(1/2, \pi, r_8) = \pm 1$. Moreover this root number is product of local root numbers $\epsilon(1/2, \pi_v, r_8) = \pm 1$. Thus we have

$$\epsilon(1/2, \pi, r_8) = (-1)^{\#\Sigma}$$

where Σ is the set of places where the local root number is -1.

In the following we assume that F is totally real with adles $\mathbb{A}$ by abuse of notation, that each π_i has parallel weight 2, and that functional equation has -1 sign. Then Σ is odd and includes all archimedean places. Then we have an incoherent quaternion algebra $\mathbb{B}$ over $\mathbb{A}$ with ramification set Σ, and Shimura curves X_U parameterized by open compact subgroup of U of $\mathbb{B}_f^\times$. The projective limit of such curves has an action by $\mathbb{B}^\times$. We write $X_E = X^3$ the triple product of such system as projective limit of $X_{U_E} = X_U \times X_U \times X_U$ with an action by $\mathbb{B}_E^\times = (\mathbb{B}^\times)^3$. We consider X as a curve in X_E via diagonal embedding. Then on each finite level $X_U \longrightarrow X_{U_E}$ where $U_E = U^3$, we can define the modified Gross–Schoen cycle $X_{U,\xi}$ using the Hodge class. Such a class form a projective system and thus defined a class in

$$X_\xi \in \mathrm{Ch}_1(X_E)^0 := \varprojlim_U \mathrm{Ch}^2(X_{U_E})^0$$

where the supscript 0 means homological trivial cycles.

Let $f \in C_0^\infty(\mathbb{B}_E^\times / D_E, \mathbb{C})$ where $D_E = D^3$, then we can define correspondence

$$Z(f) \in \mathrm{Ch}^3(X_E \times X_E) := \varinjlim_{U_E} \mathrm{Ch}^3(X_{U_E} \times X_{U_E}).$$

Thus we can define a co-cycle

$$T(f) X_\xi \in \mathrm{Ch}^2(X_E)^0.$$

The height pairing on each level X_{U_E} defines a pairing on their limit. Thus we have a well-defined number

$$\langle T(f) X_\xi, X_\xi \rangle.$$

We denote this functional as BB because of Beilinson–Bloch height pairing. With respect to the action $\mathbb{B}_E^\times \times \mathbb{B}_E^\times$ by left and right translation, we see that BB is invariant under the diagonal $\mathbb{B}^\times \times \mathbb{B}^\times$.

It has been proved by Gross–Schoen that this distribution factors though the action on weight 2 form i.e., the quotient:

$$C_0^\infty(\mathbb{B}_E^\times / D, \mathbb{C}) \longrightarrow \oplus \sigma_E \otimes \widetilde{\sigma}_E$$

where σ_E runs through the set of Jacquet–Langlands correspondences of cusp forms of parallel weight 2. In this way, we have constructed for each σ a pairing

$$\mathrm{BB}_\sigma \in \mathrm{Hom}_{\mathbb{B}^\times}(\sigma_E, \mathbb{C}) \otimes \mathrm{Hom}_{\mathbb{B}^\times}(\sigma_E, \mathbb{C}).$$

By local theory the right hand side is at most one dimensional and not zero if and only if the condition that the central character σ_E is trivial on the diagonal $\mathbb{A}^\times$. In particular for $\sigma = \pi^{JL}$, this space is one dimensional. Also, one may construct a concrete base $\beta\beta_{\pi^{JL}}$ in the space by regularize the integral

$$v \otimes \widetilde{v} \mapsto \int_{\mathbb{A}^\times \backslash \mathbb{B}^\times} (\sigma_E(g)v, \widetilde{v}) dg.$$

Theorem 5.2.1 (Yuan, Zhang, Zhang)**.**

$$\mathrm{BB}_{\pi^{JL}} = L'(1/2, \pi, r_8) \beta\beta_{\pi^{JL}}.$$

5.3 Application to elliptic curves

Assume that $F = \mathbb{Q}$ and three cusp forms π_i corresponding elliptic curves E_i, i.e., each π_i is generated by newform f_i corresponding to E_iThus π_i. Assume for simplicity that all f_i has the same conductor N which is square free. Let $f_i = \sum_{a_{in}} q^n$ are Fourier expansion. Then Σ has a description as follows

$$\Sigma = \{\infty\} \cup \{p|N : \quad a_{1p}a_{2p}a_{3p} = -1\}.$$

If Σ is odd which is equivalent to that product of signs of functional equations of f_i is -1, then there is a quaternion algebra over $\mathbb{Q}$ with ramification set $\Sigma \setminus \{\infty\}$. Let R be an order of B with reduced discriminant N, then we have a Shimura curve

$$X = R_+^\times \backslash \mathscr{H}.$$

By our construction, every E_i is uniformized by $p_i : X \longrightarrow E_i$ thus there is a morphism $X \longrightarrow E_1 \times E_2 \times E_3$. The image with modification using base point 0 on E_i will give a cycle

$$X_0 \in \mathrm{Ch}^2(E_1 \times E_2 \times E_3)^0.$$

Our main theorem in the last section gives

$$h(X_0) = L'(2, f_1 \times f_2 \times f_3) c(f_1, f_2, f_3)$$

where $c(f_1, f_2, f_3)$ is a positive and explicit constant.

There is not much can be said when all f_i are different. If all f_i are same, then X_0 is simply the Gross–Schoen cycle in E^3 thus is torsion. On the L-function part, $L(s, f \times f \times f)$ equals to $L(s, \mathrm{Sym}^3 f) L(s-1, f)^2$ which has vanishing order ≥ 1 at $s = 2$ as $L(s, f)$ has odd functional equation.

The next case is when $f_1 = f_2 \neq f_3$. Then we can construct a rational point x by the following formula

$$x = [p_{3*} p_1^*(0)] \in E_3(\mathbb{Q})$$

where [] means taking sum using group law in E_3. Then one can show that up to a positive explicit constant,

$$h(x) = h(X_0).$$

On the other hand

$$L(s, f_1 \times f_1 \times f_3) = L(s, \mathrm{Sym}^2 f_1 \times f_3) L(s-1, f_3)$$

thus we have a formula up to an explicit positive constant.

$$h(x) = L(2, \mathrm{Sym}^2 f_1 \times f_3) L'(1, f_3).$$

This formula shows that we have a cheap construction of rational points on elliptic curves using another elliptic curve.

References

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