

Serre's Conjecture and Generalizations
Advanced Course at CRM, Barcelona
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The plan for the course is as follows:

1. Statement of Serre's Conjecture

Serre's conjecture [21], now a Theorem of Khare-Wintenberger [16], [17], states that every continuous, odd, irreducible two-dimensional representation of $G_{\mathbb{Q}}$ arises from modular forms. In this lecture (and the first part of my lecture on 21 June) I'll recall the precise statement, including Serre's recipe for the minimal level and weight of the modular form. I'll also discuss a refinement of the weight part of the conjecture with a view towards its generalizations.

2. Proof of Serre's conjecture

This will be the content of Dieulefait's lectures on 17 June (16:00), 24 June (11:30) and 25 June (11:30). Here is his outline:

- Review of the MLT that will be used (results of Taylor-Wiles, Skinner-Wiles, Diamond, Kisin). Review of results of Bckle on presentations of universal global deformation rings.
- Review of results of Tate, Serre, Fontaine, Schoof and Brumer-Kramer on Galois representations and abelian varieties of small prime conductor.
- Consequences of potential modularity (results of Dieulefait and Khare- Wintenberger): existence of compatible systems, existence of Galois conjugates, and existence of lifts with prescribed properties; proof of the first cases of the Fontaine-Mazur and Serre's conjectures
- Induction on the weight (Khare): proof of the level 1 case. An alternative proof (by Dieulefait) via Galois conjugation.
- Proof of the general case (Khare-Wintenberger): new tools: good-dihedral primes, 2-adic MLT, and more weight induction.
- An alternative proof of the odd level case (Dieulefait): new tool: pseudo Sophie Germain primes and killing ramification (via results of Kisin, Caruso and Schoof)

3. **Serre's conjecture for GL_2 over totally real fields**

My other lectures will be on generalizations and refinements of Serre's conjecture, especially the case of GL_2 over a totally real field F , the main difficulty being the specification of the weight. I'll explain the statement of the conjecture in the case where p is unramified in F [3], generalizations of Schein [19] and Gee to the ramified case, and results on the weight part of the conjecture. This will include Gee's method [11] via modularity of lifts of prescribed type in the unramified case, and results of Gee-Savitt [12] via weight-cycling in the ramified case.

4. **Langlands correspondences mod p** After reviewing some of the framework for Langlands program for GL_2 , I'll describe how Serre's conjecture can be viewed as part of a "mod p Langlands program," with the specification of level and weight being corollaries of a local-global compatibility statement made precise by work of Emerton [8].

5. **Serre's conjecture for GL_n** In any remaining time, I'll discuss the statement of generalizations to the context of automorphic representations of GL_n , following Herzig [14], Gee and Savitt.

1 Statement of Serre's Conjecture

Lecture 1, 14 June 2010

Recall that the space $S_k(\Gamma_1(N))$ of cusp forms of weight k with respect to $\Gamma_1(N)$ is acted on by the Hecke operators T_n for all $n \geq 1$, and by the group $(\mathbb{Z}/N\mathbb{Z})^\times$. If

$$f(\tau) = \sum_{n \geq 1} a_n e^{2\pi i n \tau}$$

is an eigenform for all T_n , then $(\mathbb{Z}/N\mathbb{Z})^\times$ acts on f via some character ψ , and $a_1 \neq 0$, so we can normalize f so $a_1 = 1$. We then have $T_n f = a_n f$ for all $n \geq 1$, and $\langle d \rangle f = \psi(d) f$ for all $d \in (\mathbb{Z}/N\mathbb{Z})^\times$. Fix a prime p and embeddings $\overline{\mathbb{Q}} \rightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_p$. Since the eigenvalues a_n lie in a number field $E_f \subset \mathbb{C}$, we can also view $a_n \in \overline{\mathbb{Q}}_p$ via these embeddings. Let us also fix embeddings $\overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_q$ for all primes q and so view each $G_q = G_{\mathbb{Q}_q}$ as a subgroup of $G_{\mathbb{Q}}$, and write I_q for the inertia subgroup of G_q .

Constructions of Eichler-Shimura ($k = 2$), Deligne ($k > 2$) and Deligne-Serre ($k = 1$) give a continuous representation

$$\rho_f : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

such that if $q \nmid Np$, then ρ_f is unramified at q and $\rho_f(\mathrm{Frob}_q)$ has characteristic polynomial $X^2 - a_q X + \psi(q)q^{k-1}$ (where Frob_q is an arithmetic Frobenius element in G_q/I_q). Moreover $\rho_f|_{G_p}$ is de Rham (in fact crystalline of $p \nmid N$) with Hodge-Tate weights $\{0, k-1\}$. Note that $\det \rho_f = \psi \chi_p^{k-1}$ (where χ_p denotes the p -adic cyclotomic character), and ρ_f is *odd* in the sense that if c is a complex conjugation, then $\det \rho_f(c) = -1$.

To define the reduction of ρ_f , first choose a basis so that ρ_f takes values in $\mathrm{GL}_2(\overline{\mathbb{Z}}_p)$. (The easiest way to see this is possible is to use that ρ_f takes values in the finite extension $\mathbb{Q}_p(\{a_n\})$ of \mathbb{Q}_p , and therefore stabilizes a lattice since $G_{\mathbb{Q}}$ is compact.) Now define

$$\bar{\rho}_f : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

as the composite of ρ_f with the reduction map $\mathbb{Z}_p \rightarrow \mathbb{F}_p$. The Brauer-Nesbitt Theorem shows that $\bar{\rho}_f$ is well-defined up to semi-simplification; in particular it is well-defined if it is irreducible.

Serre's conjecture states that every continuous, odd, irreducible

$$\rho : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

arises this way. More precisely, given such a ρ , Serre defined integers $k(\rho)$ and $N(\rho)$ and conjectured the following, now a theorem of Khare-Wintenberger:

Theorem 1.1 $\rho \sim \bar{\rho}_f$ for some eigenform $f \in S_{k(\rho)}(N(\rho))$.

Next we recall the definitions of $N(\rho)$ and $k(\rho)$.

1.1 The level

Serre defines $N(\rho)$ as the prime-to- p Artin conductor of ρ , i.e.,

$$N(\rho) = \prod_q q^{n_q},$$

where n_q is the conductor exponent of $\rho|_{G_q}$, whose definition we now recall for an arbitrary continuous representation $\sigma : G_q \rightarrow \text{Aut} V$ where V is finite-dimensional over $\overline{\mathbb{F}}_p$ (see [20]). Let K be the fixed field of $\ker(\sigma)$, let $G = \sigma(G_q) = \text{Gal}(K/\mathbb{Q}_q)$, and define

$$G_i = \{ g \in G \mid g(x) \equiv x \pmod{\pi_K^{i+1}} \text{ for all } x \in \mathcal{O}_K^\times \}.$$

Then $G_0 = \sigma(I_q) \supset G_1 = \sigma(I_q^w) \supset G_2 \supset \dots$, and $G_i = \{1\}$ for $i \gg 0$. Artin then showed that

$$n_q := \sum_{i \geq 0} \frac{\dim(V/V^{G_i})}{[G_0 : G_i]} \in \mathbb{Z}.$$

Note that $n_q = 0$ if and only if $G_0 = \{1\}$ if and only if σ is unramified. Returning to the definition of $N(\rho)$, we see that ρ is unramified at q if and only if $N(\rho)$ is prime to q . Thus the statement of Serre's Conjecture includes Ribet's "level-lowering" theorem [18]. We remark also that for an eigenform f of level N , Carayol's theorem [5] implies that $n_q(\bar{\rho}_f) \leq v_q(N)$, providing a sense in which $N(\rho)$ is "minimal."

1.2 The weight

Recall that $N(\rho)$ only depends on the restrictions $\rho|_{D_q}$ for $q \neq p$. We will define $k(\rho)$ depending only on the $\rho|_{D_p}$. (In fact in each case the dependence is only on restrictions to inertia subgroups.)

First recall the following explicit description of the structure of G_p/I_p^w . For each $n \geq 1$, let $\pi_n = \sqrt[n]{p}$ and consider the extensions $\mathbb{Q}_p \subset K_n \subset L_n$ where $K_n = \mathbb{Q}_p(\zeta_{p^n-1})$ and $L_n = K_n(\pi_n)$. We then have the isomorphisms

$$\mathrm{Gal}(K_n/\mathbb{Q}_p) \cong \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) = \langle \mathrm{Frob}_p \rangle \cong \mathbb{Z}/n\mathbb{Z},$$

and $\mathrm{Gal}(L_n/K_n) \cong \mathbb{F}_{p^n}^\times$ via

$$g \mapsto g(\pi_n)/\pi_n \bmod \pi_n,$$

where \mathbb{F}_{p^n} is identified with the residue field of L_n (or K_n). Moreover $\mathrm{Gal}(L_n/\mathbb{Q}_p)$ is isomorphic to the semidirect product

$$\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \ltimes \mathbb{F}_{p^n}^\times.$$

Taking the direct limit over all n gives the maximal unramified extension $\mathbb{Q}_p^{\mathrm{unr}} = \bigcup_{n \geq 1} K_n$ and the maximal tamely ramified extension $\bigcup_{n \geq 1} L_n$ of \mathbb{Q}_p , so that

$$\begin{aligned} G_p/I_p &= \varprojlim \mathrm{Gal}(K_n/\mathbb{Q}_p) = \varprojlim (\mathbb{Z}/n\mathbb{Z}) = \hat{\mathbb{Z}}, \\ I_p/I_p^w &= \varprojlim \mathrm{Gal}(L_n/K_n) = \varprojlim \mathbb{F}_{p^n}^\times \cong \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \\ \text{and } G_p/I_p^w &\cong \hat{\mathbb{Z}} \ltimes \prod_{\ell \neq p} \mathbb{Z}_\ell(1). \end{aligned}$$

We define the *fundamental character of niveau n* as the composite of the natural surjective maps

$$\omega_n : I_p \rightarrow I_p/I_p^w \rightarrow \mathrm{Gal}(L_n/K_n) \cong \mathbb{F}_{p^n}^\times.$$

We can now state the crucial lemma needed in order to define $k(\rho)$.

Lemma 1.2 *Let $\rho : G_p \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}_p})$ be a continuous homomorphism.*

1. *If ρ is reducible, then $\rho|_{I_p} \sim \begin{pmatrix} \omega^a & * \\ 0 & \omega^b \end{pmatrix}$ for some $a, b \in \mathbb{Z}$ with $1 \leq a \leq p-1$ and $0 \leq b \leq p-2$.*
2. *If ρ is irreducible, then $\rho|_{I_p} \sim \begin{pmatrix} \omega_2^{a+pb} & 0 \\ 0 & \omega_2^{b+pa} \end{pmatrix}$ for some a, b as above with $a \neq b$.*

Proof.

1. If ρ is reducible, then $\rho \sim \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}$ for some $\psi_i : G_p \rightarrow \overline{\mathbb{F}}_p^\times$ for $i = 1, 2$. Note that ψ_i factors through $(G_p/I_p^w)^{\text{ab}}$, and the above description of G_p/I_p^w (or local class field theory) identifies the image of I_p in $(G_p/I_p^w)^{\text{ab}}$ with $\omega(I_p) = \mathbb{F}_p^\times$. Therefore $\psi_i|_{I_p} = \omega^{m_i}$ for some $m_i \in \mathbb{Z}$, which we can take to be in $[1, p-1]$ for $i = 1$ and $[0, p-2]$ for $i = 2$.
2. Suppose now that ρ is irreducible. We note first that $\rho(I_p^w)$ must be trivial for the following reason. Since $\rho(I_p^w)$ has p -power order, it must be conjugate to a subgroup of

$$U = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \overline{\mathbb{F}}_p \right\}.$$

Since I_p^w is normal in G_p , we must have that $\rho(G_p)$ is contained in the normalizer of $\rho(I_p^w)$. If $\rho(I_p^w)$ is non-trivial, this normalizer is contained in the group of upper-triangular matrices, which implies that ρ is reducible.

We now have that $\rho|_{I_p}$ factors through I_p/I_p^w , so $\rho(I_p)$ is abelian and of order prime to p . Therefore $\rho|_{I_p} \sim \begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix}$ for some characters $\psi_i : I_p \rightarrow \overline{\mathbb{F}}_p^\times$. Choose $g \in G_p$ lifting Frob_p , and $h \in I_p$ lifting a topological generator of I_p/I_p^w . Then $\rho(ghg^{-1}) = \rho(h^p)$ is conjugate to $\rho(h)$, but different from $\rho(h)$ (else the image of ρ would be abelian, so ρ would be reducible). It follows that $\psi_1^p = \psi_2$, $\psi_2^p = \psi_1$, $\psi_1 \neq \psi_2$. Therefore $\psi_1^{p^2-1} = 1$, so we can write $\psi_1 = \omega_2^m$ for some integer $m \in [1, p^2-2]$ not divisible by $p+1$, and it follows easily that $\rho|_{I_p}$ has the required form.

Remark 1.3 The proof also shows that if ρ is irreducible, then $\rho \sim \text{Ind}_{G_K}^{G_p} \psi$ for some character $\psi : G_K \rightarrow \overline{\mathbb{F}}_p^\times$ where K is the unramified quadratic extension of \mathbb{Q}_p .

We are now ready to define the weight in Serre's Conjecture. For ρ is an in the lemma, we set

$$k(\rho) = 1 + \max\{a, b\} + p \min\{a, b\},$$

except in the following two reducible cases:

- If $a = p - 1$, $b > 0$ and $\rho|_{I_p} \sim \begin{pmatrix} 1 & 0 \\ 0 & \omega^b \end{pmatrix}$, then we could instead have written $\rho|_{I_p} \sim \begin{pmatrix} \omega^b & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \omega^{a'} & 0 \\ 0 & \omega^{b'} \end{pmatrix}$ with $a' = b$ and $b' = 0$, giving $k(\rho) = b + 1$. (In general, the expression for ρ as in the lemma is not unique, but the formula for $k(\rho)$ is independent of that expression, except in this one case.)
- If $a = b + 1$ and $\rho \sim \psi \begin{pmatrix} \chi_p & t \\ 0 & 1 \end{pmatrix}$ is très ramifiée, then we let

$$k(\rho) = (b + 1)(p + 1) = 1 + a + pb + (p - 1).$$

Here χ_p is the mod p cyclotomic character on G_p , and *très ramifiée* means that the class defined by t in

$$H^1(G_p, \overline{\mathbb{F}}_p(\chi_p)) \cong H^1(G_p, \mu_p) \otimes \overline{\mathbb{F}}_p \cong \mathbb{Q}_p^\times \otimes \overline{\mathbb{F}}_p$$

(the last isomorphism being the Kummer map) is not in the image of $\mathbb{Z}_p^\times \otimes \overline{\mathbb{F}}_p$.

Note in particular that $2 \leq k(\rho) \leq p^2 - 1$ in all cases, and that $k(\rho) \leq p + 1$ if and only if

$$\rho|_{I_p} \sim \begin{pmatrix} \omega^a & * \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} \omega_2^a & 0 \\ 0 & \omega_2^{pa} \end{pmatrix} \text{ for some } a \in 1, \dots, p - 1.$$

Moreover note we then have $k(\rho) = 1 + a$ unless ρ is très ramifiée, in which case $k = 1 + p$. Note furthermore that for any ρ , there is an integer m such that $\rho \otimes \chi_p^{-m}$ has the above form, so that

$$k(\rho \otimes \chi_p^{-m}) \leq p + 1.$$

We can now motivate the definition for $k(\rho)$, at least in the cases where $k(\rho) \leq p + 1$, with the following theorem of Deligne and Fontaine (see [7]):

Theorem 1.4 *Suppose that f is an eigenform in $S_k(\Gamma_1(N))$ with $2 \leq k \leq p + 1$ and $p \nmid N$. Then*

$$\overline{\rho}_f|_{I_p} \sim \begin{pmatrix} \omega^{k-1} & * \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} \omega_2^{k-1} & 0 \\ 0 & \omega_2^{p(k-1)} \end{pmatrix}.$$

Moreover if $\overline{\rho}_f$ is très ramifiée, then $k = p + 1$.

1.3 Hecke operators and cohomology

We give now a characterization of being “modular of weight k and level N ” in terms of the Hecke action on cohomology.

For a finite set of primes S , we define the algebra \mathbb{T}^S as the polynomial ring over \mathbb{Z} with variables T_q and $\langle q \rangle$ for all primes $q \notin S$. Let $\mathbb{T} = \mathbb{T}^S$ for a set of primes S containing those dividing Np and consider the action of \mathbb{T} on $S_k(\Gamma_1(N))$, and let $I = \text{Ann}_{\mathbb{T}} S_k(\Gamma_1(N))$.

If f is an eigenform, then denote by θ_f the algebra homomorphism $\mathbb{T} \rightarrow \overline{\mathbb{Z}}_p$ defined by $T_q \mapsto a_q$ and $\langle q \rangle \mapsto \psi(q)$, and let $\mathfrak{p}_f = \ker \theta_f$. Since $Tf = \theta_f(T)f$, it follows that $\mathfrak{p}_f \supset I$. Conversely, one can show that if $\mathfrak{p} \subset \mathbb{T}$ is any prime ideal such that $\mathfrak{p} \cap \mathbb{Z} = \{0\}$ and $\mathfrak{p} \supset I$, then $\mathfrak{p} = \mathfrak{p}_f$ for some such eigenform f . (In fact the set of such \mathfrak{p} is in bijection with the set of Galois conjugacy classes of newforms f of weight k and level dividing N .)

We can similarly consider the homomorphism $\bar{\theta}_f : \mathbb{T} \rightarrow \overline{\mathbb{F}}_p$ gotten by composing θ_f with reduction, and let $\mathfrak{m}_f = \ker \bar{\theta}_f$. Then any maximal ideal $\mathfrak{m} \subset \mathbb{T}$ containing (p, I) is such an \mathfrak{m}_f . (For this, apply the Going Down Theorem to the integral extension \mathbb{T}/I of \mathbb{Z} .)

Suppose now that $\rho : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$ and that S is also contains the set of primes at which ρ is ramified. We then define $\theta_\rho : \mathbb{T} \rightarrow \overline{\mathbb{F}}_p$ by

$$T_q \mapsto \text{tr}(\rho(\text{Frob}_q)), \quad \langle q \rangle \mapsto q^{1-k} \det(\rho(\text{Frob}_q)),$$

and let $\mathfrak{m}_\rho = \ker \theta_\rho$. Since ρ is determined by its characteristic polynomials, we see that $\rho \sim \bar{\rho}_f$ if and only if $\theta_\rho = \bar{\theta}_f$, which implies that $\mathfrak{m}_\rho = \mathfrak{m}_f$. Conversely if $\mathfrak{m}_\rho = \mathfrak{m}_f$, we see that $\theta_\rho = \sigma \circ \bar{\theta}_f$ for some $\sigma \in G_{\overline{\mathbb{F}}_p}$, so that $\theta_\rho = \theta_g$ for some g Galois conjugate to f . We thus conclude that ρ is modular of weight k and level N if and only if $\mathfrak{m}_\rho \supset I$.

We also have an action of \mathbb{T} on the group cohomology $H^1(\Gamma_1(N), \text{Sym}^{k-2}\mathbb{Z}^2)$ defined in a standard way (the Hecke operator is a composite of restriction and trace maps determined by the corresponding double coset). In fact the action is defined on $H^1(\Gamma_1(N), \text{Sym}^{k-2}R^2)$ for any ring R and we have the following:

Lemma 1.5 *If $p \nmid N$, then the following are equivalent:*

- ρ is modular of weight k and level N ;
- \mathfrak{m}_ρ is in the support of $H^1(\Gamma_1(N), \text{Sym}^{k-2}\mathbb{Z}_p^2)$;

- \mathfrak{m}_ρ is in the support of $H^1(\Gamma_1(N), \text{Sym}^{k-2}\mathbb{F}_p^2)$;
- \mathfrak{m}_ρ is in the support of $H^1(\Gamma_1(N), V)$ for some $\text{GL}_2(\mathbb{F}_p)$ -Jordan-Hölder constituent V of $\text{Sym}^{k-2}\mathbb{F}_p^2$.

We briefly give the idea of the proof. We have already seen that the first condition is equivalent to $\mathfrak{m}_\rho \supset I$. The Eichler-Shimura isomorphism gives a \mathbb{T} -equivariant isomorphism between $S_k(\Gamma_1(N))$ and $H_{\text{par}}^1(\Gamma_1(N), \text{Sym}^{k-2}\mathbb{R}^2)$, so I is also the annihilator in \mathbb{T} of $H_{\text{par}}^1(\Gamma_1(N), \text{Sym}^{k-2}\mathbb{Z}^2)_{\text{tf}}$ (where par denotes parabolic cohomology and tf the maximal torsion-free quotient). The rest of the proof consists of writing down exact sequences relating the relevant cohomology groups and checking that various kernels and cokernels are “Eisenstein.” The irreducibility of ρ then implies that \mathfrak{m}_ρ cannot be in the support of these “error terms.”

Some remarks about the Jordan-Hölder factors V are in order.

- The action of $\Gamma_1(N)$ on $\text{Sym}^{k-2}\mathbb{F}_p^2$ is via the reduction map $\Gamma_1(N) \rightarrow \text{SL}_2(\mathbb{F}_p)$, but the definition of the Hecke action requires the action of $\text{GL}_2(\mathbb{F}_p)$.
- The irreducible representations of $\text{GL}_2(\mathbb{F}_p)$ over \mathbb{F}_p are precisely the representations

$$V_{m,n} = \det^m \otimes \text{Sym}^n \mathbb{F}_p^2$$

for integers m, n satisfying $0 \leq m \leq p-2$, $0 \leq n \leq p-1$.

- \mathfrak{m}_ρ is in the support of $H^1(\Gamma_1(N), V_{m,n})$ if and only if $\mathfrak{m}_{\rho \otimes \chi_p^{-m}}$ is in the support of $H^1(\Gamma_1(N), V_{0,n})$. Indeed we have $V_{m,n} = V_{0,n}$ as $\text{GL}_2(\mathbb{F}_p)$ -modules, but the action of T_q (resp. $\langle q \rangle$) on $H^1(\Gamma_1(N), V_{0,n})$ is q (resp. q^2) times its action on $H^1(\Gamma_1(N), V_{m,n})$.
- One can easily compute the Jordan-Hölder factors of $\text{Sym}^n \mathbb{F}_p^2$ for $n \geq p$. Indeed using Brauer characters we see that the semisimplification of $\text{Sym}^p \mathbb{F}_p^2$ is $\text{Sym}^1 \mathbb{F}_p^2 \oplus \det \text{Sym}^{p-1} \mathbb{F}_p^2$, and that if $n \geq p+1$, then $\text{Sym}^n \mathbb{F}_p^2$ has the same semisimplification as

$$\det \text{Sym}^{n-p-1} \mathbb{F}_p^2 \oplus \text{Sym}^{[n]} \mathbb{F}_p^2 \oplus \det^n \text{Sym}^{p-1-[n]} \mathbb{F}_p^2$$

where $[n] \equiv n \pmod{p-1}$ and $0 \leq [n] \leq p-2$.

From the preceding lemma and remarks, we get the following result of Ash and Stevens [1]:

Corollary 1.6 *If ρ is modular of level N and weight k , then $\rho \otimes \chi_p^m$ is modular of level N and weight k' for some integers m, k' with $2 \leq k' \leq p+1$.*

We have given the ingredients of the proof in the case $p \nmid N$ and $k \geq 2$, but done more carefully, one can in fact allow level Np^r and weight $k \geq 1$ in the assumptions and obtain the same conclusion.

1.4 Serre weights

Lecture 2, 21 June 2010

Lemma 1.5 and the subsequent remarks motivate the following definition (see [15]) for a continuous, odd, irreducible $\rho : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ and integers m, n, N such that $0 \leq m \leq p-2$, $0 \leq n \leq p-1$ and $p \nmid N$.

Definition 1.7 We say that ρ is *modular of weight $V_{m,n}$* and level N if $\rho \otimes \chi_p^{-m}$ is modular of weight $n+2$ and level N , or equivalently, if \mathfrak{m}_{ρ} is in the support of $H^1(\Gamma_1(N), V_{m,n})$.

Note that by Lemma 1.5, the set of $k \geq 2$ such that ρ is modular of weight k and level N is determined by the set of $V_{m,n}$ such that ρ is modular of weight $V_{m,n}$ and level N .

Given $\rho : G_p \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, we define the set of *Serre weights* of ρ as the subset $S(\rho) \subset \{V_{m,n} \mid 0 \leq m \leq p-2, 0 \leq n \leq p-1\}$ characterized by

$$V_{m,n} \in S(\rho) \Leftrightarrow \rho|_{I_p} \sim \begin{cases} \omega^m \begin{pmatrix} \omega^{n+1} & * \\ 0 & 1 \end{pmatrix} & (\text{peu ramifiée if } n=0), \\ \text{or } \omega^m \begin{pmatrix} \omega_2^{n+1} & 0 \\ 0 & \omega^{p(n+1)} \end{pmatrix} \end{cases}$$

As motivation for the definition of $S(\rho)$ and generalizations to come, note that since $\rho_f|_{G_p}$ is crystalline with Hodge-Tate weights $\{0, k-1\}$, Theorem 1.4 is a consequence of the following result of Fontaine-Laffaille [10] (for $n < p-1$) and Berger-Li-Zhu [2] (for $n = p-1$).

Theorem 1.8 *$V_{m,n} \in S(\rho)$ if and only if ρ has a crystalline lift with Hodge-tate weights $\{m, m+n+1\}$.*

Note that Theorem 1.4 says that if ρ is modular of weight V and some level prime to p , then $V \in S(\rho|_{G_p})$. Proofs of the weight part of Serre's Conjecture actually establish the following converse:

Theorem 1.9 *Suppose that ρ is modular of some weight and level Np^r . If $V \in S(\rho|_{G_p})$, then ρ is modular of weight V and level N .*

The proof of this result is composed of a companion forms theorem (for which there are now several independent proofs, due to Gross-Cais [13], [4], Coleman-Voloch [6], Faltings-Jordan [9] and Gee [11]), Mazur's Principle, and a θ - or weight-cycling argument (due respectively to Edixhoven [7] and Buzzard). Gee's method and weight-cycling will be sketched in subsequent lectures in a more general context.

Finally we remark that the computation of Jordan-Hölder factors of $\mathrm{Sym}^{k-2}\mathbb{F}_p^2$ allows one to recover Serre's $k(\rho)$ as

$$\min\{k \geq 2 \mid \mathrm{Sym}^{k-2}\mathbb{F}_p^2 \text{ has a Jordan-Hölder factor in } S(\rho|_{G_p})\}.$$

Theorem 1.9 thus has the following corollary:

Corollary 1.10 *If ρ is modular of weight k and level Np^r , then ρ is modular of weight $k(\rho)$ and level N . Moreover if $k \geq 2$ and $r = 0$, then $k(\rho) \leq k$.*

References

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